Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 209 (2007) 583–620
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

On the cobordism and commutative monoid with cancellation approaches to conformal field theory

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Received 17 November 2005; received in revised form 20 June 2006
Available online 22 September 2006
Communicated by E.M. Friedlander

Abstract

In the late 1980s, Graeme Segal axiomatized conformal field theory in terms of a cobordism category. In that same preprint he outlined a more symmetric trace approach, which was recently rigorized in terms of pseudo algebras over a 2-theory. In this paper, we treat the cobordism approach in the pseudo algebra context. We introduce a new algebraic structure on a bicategory, called a *pseudo 2-algebra over a theory*, as a means of comparison for the two approaches. The main result states that the 2-category of pseudo algebras over a fixed 2-theory is biequivalent to the 2-category of pseudo 2-algebras over a fixed theory in certain situations.

MSC: Primary: 81T40; 18D05; secondary: 18C10

1. Introduction

In the foreword to [11], Graeme Segal points out that conformal field theory with one dimensional anomaly can be axiomatized using the 'cobordism category' where objects are 1-dimensional manifolds and morphisms are worldsheets with boundary, thought of as their cobordism. Then a conformal field theory with one dimensional anomaly can be thought of as a symmetric monoidal functor from this category to Hilbert spaces (where morphisms are defined up to scalar multiple).

This seems to be substantially simpler than the older, more symmetric approach which treats all boundary components on equal footing, and uses trace instead of compositions to axiomatize gluing. This symmetric approach was in fact only outlined in [11], and was made completely rigorous in [3,5,6]. Unlike the cobordism approach, the older approach axiomatizes conformal field theory with n-dimensional modular functor. In this paper, I shall point out certain more subtle points of the relationship of these two approaches, which in the end shows that detailed treatments of both approaches are essentially equally technical.

The completely rigorous approach to conformal field theory of [3,5,6] relies on various notions of category theory. The first ingredients are the notion of Lawvere theory as in [8] and its generalization called 2-theory in [5]. Theories are used to describe algebraic structures on a set, such as a commutative monoid. On the other hand, 2-theories are

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0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2006.07.006

used to capture algebraic structures on families of sets (*i.e.* operations indexed by other operations), such as disjoint union and gluing on worldsheets. These operations are however not strictly associative and unital in the main example of worldsheets, and this is where weak 2-category (bicategory) theory enters the picture. Pseudo algebras over theories and 2-theories capture the coherence isomorphisms and coherence diagrams necessary to treat these deficiencies.

A pseudo algebra over a 2-theory is in some sense a variant of a bicategory. Variants of bicategories, and more generally variants of enriched categories, can be found in many branches of mathematics. This work is an application of enriched category theory to rigorously treat a geometric subject, which is of general mathematical interest.

The present article introduces the 2-theory of Frobenius symmetric monoidal categories and the cobordism 2-theory, as well as a new algebraic structure on a bicategory, namely a pseudo 2-algebra over a theory. These new concepts are used to relate a rigorous description of the cobordism approach to the approach in [3,5,6]. These new 2-theories and this algebraic structure on a bicategory are also of general categorical interest.

Next I would like to emphasize that the symmetric approach of [11], which is rigorized in [3,5,6] using 'stacks of lax[1] commutative monoids with cancellation', treats a more general situation, namely conformal field theory with n-dimensional modular functor. This means that boundary components can have labels and instead of having vacua defined up to scalar multiple, they are defined up to finite dimensional vector spaces (= a 'modular functor'). If we only wish to discuss the case of one dimensional modular functor (and one label), which is the case discussed in the cobordism approach in the forward to [11], then the symmetric approach also simplifies substantially and the full machinery of [3,5,6] is not needed.

On the other hand, the cobordism approach can be generalized to a setting which can handle modular functors as in Section 6, and in fact to an abstract setting, which is not restricted to the particular case of worldsheets, but axiomatizes the kind of general categorical context in which this approach works. Discussing this general setting is the main point of this paper.

The general setting of the cobordism approach, as it turns out, involves many of the same elements as the more symmetric approach of [3,5,6], in particular 2-category theory. It also requires axiomatizing a certain condition (the triangle identities (3) and (4) of Definition 3.1), which I call the 'Frobenius condition' in reference to an analogous equation that holds in any Frobenius algebra. It is satisfied in the category of worldsheets, once the relevant modifications are undertaken in Section 4. The main result of this paper then states that the resulting structure of 'stack of pseudo Frobenius symmetric monoidal categories' implies the structure of 'stack of pseudo commutative monoids with cancellation' (SPCMC[2]) of [5,6], which is the basic structure of the symmetric approach of [3,5,6]. Since we are speaking of chiral conformal field theory, the Grothendieck site for these stacks consists of the category of finite dimensional complex manifolds with the analytic topology. A collection $\{B_i \longrightarrow B\}_i$ of open holomorphic embeddings is a cover if and only if their combined image covers B. The 'stacking' of pseudo algebras is not difficult, and is described in [3,5,6]. Therefore in this article we work with sections over a point, and do not mention stacks outside of the Introduction.

Additionally, the notion of a pseudo Frobenius symmetric monoidal category has one advantage, namely that it is based on a more familiar structure of symmetric monoidal category. As a result, it is relatively easy to list its coherence diagrams explicitly (see Theorem 3.7 below). This situation is better than that of SPCMC's, where there is an abstract machine that generates the coherence diagrams, although they are tedious to check individually. A drawback of the notion of pseudo Frobenius symmetric monoidal category is that the class of worldsheets must be modified to include infinitely thin annuli to form an example. Worse yet, the notion is not entirely satisfactory for conformal field theory, because the Frobenius elements would map to morphisms of Hilbert spaces that are not Hilbert–Schmidt. We introduce the *cobordism 2-theory* and its pseudo algebras to remedy both of these problems.

This paper links the cobordism approach to conformal field theory (with modular functor) and the SPCMC approach in a general abstract setting and is organized as follows. After a description of worldsheets in Section 2, we define the notion of a strict Frobenius symmetric monoidal category in Section 3, which is the simplified strict version of the algebraic structure in the cobordism approach. A strict Frobenius symmetric monoidal category gives rise to a strict commutative monoid with cancellation (CMC) via a map of 2-theories. As noted in [2], a pseudo category is *not* a bicategory *nor* a double category. Therefore in passing from strict Frobenius symmetric monoidal categories to

[1] In [5,6] the term 'lax' is used to mean up to coherence *isos* which satisfy coherence diagrams. Category theorists prefer to use the term 'pseudo' for this concept. In the present paper we follow the category-theoretic language and use 'pseudo' instead of 'lax'.

[2] In this paper we will use the abbreviation SPCMC instead of the abbreviation SLCMC of [5,6], although they mean exactly the same thing.

pseudo Frobenius symmetric monoidal categories, we do *not* obtain a Frobenius symmetric monoidal *bicategory*. This is related to the distinguished role of the bijections. Theorem 3.7 explains how pseudo Frobenius symmetric monoidal categories are related to symmetric monoidal bicategories with a distinguished role for the morphisms in I in terms of a 2-functor $I \longrightarrow \mathcal{C}$. In Section 4 we prove that the worldsheets admit the structure of a pseudo Frobenius symmetric monoidal category. This requires a modification of the cobordism bicategory to include infinitely thin annuli. In Section 5, we introduce the *cobordism 2-theory* and its pseudo algebras, of which the unmodified worldsheets form an example. The analogue of Theorem 3.7 is Theorem 5.7, which explains how pseudo algebras over the cobordism 2-theory are related to bicategories. Every pseudo algebra over the cobordism 2-theory gives rise to a pseudo algebra over the 2-theory of commutative monoids with cancellation. We review the two approaches to conformal field theory and modular functor in Section 6 and prove that a conformal field theory with modular functor in the cobordism approach gives rise to a conformal field theory with modular functor in the commutative monoid with cancellation approach in Theorem 6.8.

Theorem 3.7 is a special case of Theorem 7.9, which shows pseudo 2-algebras over a theory T with certain adjoined operations are biequivalent to pseudo (Θ, T)-algebras. These pseudo 2-algebras, which are introduced in Section 7, provide an intermediary between pseudo algebras over a theory and pseudo algebras over a 2-theory, which in turn allows us to relate the cobordism approach and the SPCMC approach to conformal field theory.

Appendix A reviews the relevant bicategorical notions as well as pseudo algebras over theories and 2-theories as set forth in [3,5,6].

2. Motivating example: Worldsheets

The bicategory of worldsheets provides the motivating example for the algebraic structure this paper is about. A *worldsheet* is a real, compact, not necessarily connected, two dimensional, smooth manifold with complex structure and real analytically parametrized boundary components.[3] A boundary component k is called *inbound* or *outbound* depending on the orientation of its parametrization $f_k : S^1 \longrightarrow k$ with respect to the orientation on k induced by the complex structure. The convention is to call the identity parametrization of the boundary of the unit disk *inbound*.

The worldsheets fit together to form a bicategory $\mathcal{C}$ in the following way. The objects of $\mathcal{C}$ are finite sets. A morphism from a finite set A to a finite set B is a worldsheet equipped with bijections between the set of inbound respectively outbound components and A respectively B. Morphisms are composed by gluing manifolds along boundary components with the same label. A 2-cell from a morphism x to a morphism y is a holomorphic diffeomorphism which preserves the boundary parametrizations and boundary labellings. This is a variant of the category introduced in [11], which is adapted here to rigorously study all of its properties.

But what are the identity morphisms of $\mathcal{C}$? As we'll see later, technical issues arise. To solve these difficulties, there are essentially two approaches. The approach in Section 4 is to allow infinitely thin annuli and to make some other modifications so that $\mathcal{C}$ satisfies a more naive set of axioms, which we call a *pseudo Frobenius symmetric monoidal category*. We will see later in Example 5.2 that the preferred approach is to refine the algebraic structure we are considering even further so that infinitely thin annuli are not needed.

For now, let us stay with the more naive structure of a pseudo Frobenius symmetric monoidal category. There is a pseudo functor (often called homomorphism of bicategories) $+ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ given by a choice of disjoint union. We also have a unit $0 = \emptyset$ and distinguished morphisms $m_A : 0 \longrightarrow A + A$ and $n_A : A + A \longrightarrow 0$ for each $A \in \operatorname{Obj} \mathcal{C}$. Then

$$\begin{aligned}
&m_{A+B} \cong m_A + m_B \\
&n_{A+B} \cong n_A + n_B \\
&(1_A + n_A) \circ (m_A + 1_A) \cong 1_A \\
&\text{and} \quad (n_A + 1_A) \circ (1_A + m_A) \cong 1_A.
\end{aligned}$$

This algebraic structure and these coherence isomorphisms fit together in Theorem 4.1, which states that the bicategory of worldsheets (with the technical modifications of Section 4) forms a pseudo Frobenius symmetric monoidal category.

[3] The term *rigged surface* in [3,5,6] is synonymous with the term *worldsheet* in this paper. This usage of the term *rigged surface* has come to differ from the usage in the classic paper [11], so we avoid it.

3. Frobenius implies commutative monoid with cancellation

In this section we define the strict version of the algebraic structure present on the bicategory of worldsheets and describe it in terms of 2-theories. We also show how a strict commutative monoid with cancellation (the algebraic structure in [3,5,6]) can be obtained from a strict Frobenius symmetric monoidal category.

Definition 3.1. A *strict Frobenius symmetric monoidal category* $\mathcal{C}$ is a category $\mathcal{C}$ with

- a functor $+ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ that is strictly associative, strictly commutative, and strictly unital with unit 0. In other words, $\mathcal{C}$ is a strict symmetric monoidal category.
- A morphism $m_A : 0 \longrightarrow A + A$ and a morphism $n_A : A + A \longrightarrow 0$ for each $A \in \text{Obj } \mathcal{C}$ such that

$$m_{A+B} = m_A + m_B \tag{1}$$
$$n_{A+B} = n_A + n_B \tag{2}$$
$$(1_A + n_A) \circ (m_A + 1_A) = 1_A \tag{3}$$
$$\text{and} \quad (n_A + 1_A) \circ (1_A + m_A) = 1_A \tag{4}$$

  for all $A, B \in \text{Obj } \mathcal{C}$. These m_A and n_A are called *Frobenius elements* and axioms (3) and (4) are called the *Frobenius condition*.

I thank John Baez for pointing out that the Frobenius condition in a strict symmetric monoidal category is equivalent to the requirement that each object is its own *dual* as defined in [7]. Self-duality can be replaced by a weaker condition that requires the existence of duals, but then the axioms would have to be modified in subtle ways. The present definition with self-duality is sufficiently general to handle worldsheets, where self-duality does occur after the modifications of Section 4. The motivation for the terminology 'Frobenius' in Definition 3.1 was the well known fact that a 1+1 dimensional topological field theory is the same thing as a commutative Frobenius algebra. The present definition is intended to axiomatize the source category of 'worldsheets' for a general conformal field theory as in Section 4.

Definition 3.2. Let $\mathcal{C}$ and $\mathcal{C}'$ be strict Frobenius symmetric monoidal categories with the same object set I. Then a *morphism* $F : \mathcal{C} \longrightarrow \mathcal{C}'$ over I is a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ that is the identity on I and strictly preserves $+$, 0, and the Frobenius elements.

The language of 2-theories can be used to describe Frobenius symmetric monoidal categories because the operations are indexed. A strict Frobenius symmetric monoidal category is the same thing as an algebra over the *2-theory of Frobenius symmetric monoidal categories*, which has $k = 2$. See Definition A.15 of Appendix A for details of 2-theories. The underlying 1-theory of this 2-theory is the theory of commutative monoids. We describe this 2-theory in terms of its algebras as follows.

Remark 3.3. A *strict algebra* over the *2-theory of Frobenius symmetric monoidal categories* consists of a commutative monoid $(I, +)$ and a function $X : I^2 \longrightarrow \mathit{Sets}$ with natural[4] operations

$$X_{A,B} \times X_{C,D} \xrightarrow{+} X_{A+C,B+D}$$
$$\{*\} \xrightarrow{0} X_{0,0}$$
$$X_{B,C} \times X_{A,B} \xrightarrow{\circ} X_{A,C}$$
$$\{*\} \xrightarrow{1_B} X_{B,B}$$
$$\{*\} \xrightarrow{m_A} X_{0,A+A} \qquad \{*\} \xrightarrow{n_A} X_{A+A,0}$$

for all $A, B, C, D \in I$. These operations satisfy the following relations.

[4] Here the sets I and $X_{*,*}$ are regarded as *discrete* categories, *i.e.* categories in which the only morphisms are identities. As such, naturality in Remark 3.3 is automatic. Nevertheless, we include the term 'natural' because the operations of any pseudo algebra $X : I^2 \longrightarrow \mathit{Cat}$ over a 2-theory are 2-natural transformations. In a pseudo algebra over the 2-theory at hand, I is a (not necessarily discrete) symmetric monoidal category.

(1) $\circ$ is *associative*.

$$\begin{tikzcd}
(X_{C,D} \times X_{B,C}) \times X_{A,B} \arrow[r, "\circ \times 1_{X_{A,B}}"] \arrow[dd, "\cong"'] & X_{B,D} \times X_{A,B} \arrow[dr, "\circ"] & \\
& & X_{A,D} \\
X_{C,D} \times (X_{B,C} \times X_{A,B}) \arrow[r, "1_{X_{C,D}} \times \circ"'] & X_{C,D} \times X_{A,C} \arrow[ur, "\circ"'] &
\end{tikzcd}$$

(2) For each $B \in I$, the operation 1_B is an *identity* for $\circ$.

$$\begin{tikzcd}
\{*\} \times X_{A,B} \arrow[r, "1_B \times 1_{X_{A,B}}"] \arrow[dr, "pr_2"'] & X_{B,B} \times X_{A,B} \arrow[d, "\circ"] \\
& X_{A,B}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{B,C} \times \{*\} \arrow[r, "1_{X_{B,C}} \times 1_B"] \arrow[dr, "pr_1"'] & X_{B,C} \times X_{B,B} \arrow[d, "\circ"] \\
& X_{B,C}
\end{tikzcd}$$

(3) $+$ is *associative*.

$$\begin{tikzcd}
(X_{A,B} \times X_{C,D}) \times X_{E,F} \arrow[r, "+\times 1_{X_{E,F}}"] \arrow[d, "\cong"'] & X_{A+C,B+D} \times X_{E,F} \arrow[d, "+"] \\
X_{A,B} \times (X_{C,D} \times X_{E,F}) \arrow[d, "1_{X_{A,B}} \times +"'] & X_{(A+C)+E,(B+D)+F} \arrow[d, equal] \\
X_{A,B} \times X_{C+E,D+F} \arrow[r, "+"'] & X_{A+(C+E),B+(D+F)}
\end{tikzcd}$$

(4) $+$ is *commutative*.

$$\begin{tikzcd}
X_{A,B} \times X_{C,D} \arrow[r, "+"] \arrow[d, "\cong"'] & X_{A+C,B+D} \arrow[d, equal] \\
X_{C,D} \times X_{A,B} \arrow[r, "+"'] & X_{C+A,D+B}
\end{tikzcd}$$

(5) 0 is a *unit* for $+$.

$$\begin{tikzcd}
X_{A,B} \times \{*\} \arrow[r, "1_{X_{A,B}} \times 0"] \arrow[d, "pr_1"] & X_{A,B} \times X_{0,0} \arrow[d, "+"] \\
X_{A,B} \arrow[r, equal] & X_{A+0,B+0}
\end{tikzcd}
\qquad
\begin{tikzcd}
\{*\} \times X_{A,B} \arrow[r, "0 \times 1_{X_{A,B}}"] \arrow[d, "pr_2"] & X_{0,0} \times X_{A,B} \arrow[d, "+"] \\
X_{A,B} \arrow[r, equal] & X_{0+A,0+B}
\end{tikzcd}$$

(6) The *unit* $0 \in X_{0,0}$ is the *identity* 1_0 (in the sense of (2)) on $0 \in I$.

(7) m and n are *compatible* with $+$.

$$\begin{tikzcd}
\{*\}^{\times 2} \arrow[r, "m_A \times m_B"] \arrow[dd, "\cong"'] & X_{0,A+A} \times X_{0,B+B} \arrow[dr, "+"] & \\
& & X_{0+0,(A+A)+(B+B)} \\
\{*\} \arrow[r, "m_{A+B}"'] & X_{0,(A+B)+(A+B)} \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
\{*\}^{\times 2} \arrow[r, "n_A \times n_B"] \arrow[dd, "\cong"'] & X_{A+A,0} \times X_{B+B,0} \arrow[dr, "+"] & \\
& & X_{(A+A)+(B+B),0+0} \\
\{*\} \arrow[r, "n_{A+B}"'] & X_{(A+B)+(A+B),0} \arrow[ur, equal] &
\end{tikzcd}$$

(8) The *Frobenius axiom* holds.

$$\begin{tikzcd}
\{*\}^{\times 4} \arrow[r, "(n_A \times 1_A)\times(1_A \times m_A)"] \arrow[ddd, "\cong"'] & (X_{A+A,0} \times X_{A,A}) \times (X_{A,A} \times X_{0,A+A}) \arrow[d, "+\times+"] \\
& X_{(A+A)+A,0+A} \times X_{A+0,A+(A+A)} \arrow[d, equal] \\
& X_{(A+A)+A,A} \times X_{A,(A+A)+A} \arrow[d, "\circ"] \\
\{*\} \arrow[r, "1_A"'] & X_{A,A}
\end{tikzcd}$$

$$\begin{tikzcd}
\{*\}^{\times 4} \arrow[r, "(1_A \times n_A)\times(m_A \times 1_A)"] \arrow[ddd, "\cong"'] & (X_{A,A} \times X_{A+A,0}) \times (X_{0,A+A} \times X_{A,A}) \arrow[d, "+\times+"] \\
& X_{A+(A+A),0+A} \times X_{0+A,(A+A)+A} \arrow[d, equal] \\
& X_{A+(A+A),A} \times X_{A,A+(A+A)} \arrow[d, "\circ"] \\
\{*\} \arrow[r, "1_A"'] & X_{A,A}.
\end{tikzcd}$$

Lemma 3.4. *A strict algebra over the 2-theory of Frobenius symmetric monoidal categories is precisely the same as a strict Frobenius symmetric monoidal category as in Definition 3.1. A morphism of strict algebras over the 2-theory of strict Frobenius symmetric monoidal categories with same underlying I is the same as a morphism of Frobenius symmetric monoidal categories over I as in Definition 3.2.*

Recall that a *strict commutative monoid with cancellation* is a strict algebra over the *2-theory of commutative monoids with cancellation* as in Example A.17 of Appendix A.

Theorem 3.5. *A strict Frobenius symmetric monoidal category $\mathcal{C}$ gives rise to a strict commutative monoid with cancellation $X : I^2 \longrightarrow Sets$ whose underlying strict commutative monoid is $I = \text{Obj } \mathcal{C}$.*

Proof. Suppose $\mathcal{C}$ is a strict Frobenius symmetric monoidal category. Define a function $X : I^2 \longrightarrow Sets$ by $X_{A,B} := \text{Mor}_{\mathcal{C}}(A, B)$. The operation

$$+ : X_{A,B} \times X_{C,D} \longrightarrow X_{A+C,B+D}$$

is the restriction of $+ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ to the relevant hom sets. To define 'gluing' $\check{?} : X_{A+C,B+C} \longrightarrow X_{A,B}$ we use the auxiliary operation

$$X_{A,B+C} \xrightarrow{s_C} X_{A+C,B}$$
$$f \longmapsto (1_B + n_C) \circ (f + 1_C).$$

Note that $s_C \circ s_D = s_{C+D} = s_{D+C} = s_D \circ s_C$. The gluing is the composition

$$\check{?} : X_{A+C,B+C} \xrightarrow{s_C} X_{A+C+C,B} \xrightarrow{\circ(1_A+m_C)} X_{A,B}.$$

The unit $0 \in X_{0,0}$ is the identity on the zero of I in $\text{Mor}_{\mathcal{C}}(0, 0)$. □

Theorem 3.6. *There is a natural morphism of 2-theories over the theory of commutative monoids from the 2-theory of commutative monoids with cancellation to the 2-theory of Frobenius symmetric monoidal categories.*

Proof. The 'definitions' in the proof of 3.5 actually describe where the abstract words $+$, $\check{?}$, and 0 of the 2-theory of commutative monoids with cancellation map to in the 2-theory of Frobenius symmetric monoidal categories. □

Now that we have defined the 2-theory of pseudo Frobenius symmetric monoidal categories, we can speak of pseudo algebras over this 2-theory, *i.e.* pseudo Frobenius symmetric monoidal categories. See Definition A.19 of Appendix A for the notion of pseudo algebra over a 2-theory. In this terminology, we can also say that Theorem 3.6 induces a forgetful 2-functor of the pseudo structures, *i.e.* a pseudo Frobenius symmetric monoidal category gives rise to a pseudo commutative monoid with cancellation. This forgetful 2-functor admits a left biadjoint by an argument similar to Chapter 10 of [3], which contains a proof that forgetful 2-functors of pseudo algebras over theories admit left biadjoints. The 2-category of pseudo Frobenius symmetric monoidal categories also admits weighted pseudo limits by Theorem 13.11 of [3].

The advantage of using pseudo Frobenius symmetric monoidal categories however is that they are familiar structures and the coherences are easily identified.

Theorem 3.7. *The 2-category of pseudo Frobenius symmetric monoidal categories with underlying groupoid I and strict identity morphisms is biequivalent to the 2-category of bicategories $\mathcal{C}$ equipped with an operation*

$$+ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C},$$

an object $0 \in \text{Obj } \mathcal{C}$, morphisms $m_A \in \text{Mor}_{\mathcal{C}}(0, A + A)$ and $n_A \in \text{Mor}_{\mathcal{C}}(A + A, 0)$, and a groupoid I with strict 2-functors $P : I \longrightarrow \mathcal{C}$ and $+ : I \times I \longrightarrow I$ satisfying:

- *$+ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a pseudo functor (homomorphism of bicategories).*
- *The 2-functor $P : I \longrightarrow \mathcal{C}$ is the identity on objects, $\text{Obj } I = \text{Obj } \mathcal{C}$, and $P(A +_I B) = A +_{\mathcal{C}} B$.*
- *$A \mapsto m_A$ and $A \mapsto n_A$ are natural transformations for morphisms in I.*
- *$(I, +)$ is a symmetric monoidal category with coherence isos*

$$\mathfrak{a}_{A,B,C} : A + (B + C) \xrightarrow{\cong} (A + B) + C \qquad \mathfrak{c}_{A,B} : A + B \xrightarrow{\cong} B + A$$
$$\mathfrak{l}_A : 0 + A \xrightarrow{\cong} A \qquad \mathfrak{r}_A : A + 0 \xrightarrow{\cong} A.$$

This means that the following six diagrams from Mac Lane strictly commute for all $A, B, C,$ and D in $\text{Obj } I = \text{Obj } \mathcal{C}$.

(1)

$$\begin{tikzcd}
A(B(CD)) \arrow[r, "1_A \mathfrak{a}_{B,C,D}"] \arrow[d, "\mathfrak{a}_{A,B,(CD)}"'] & A((BC)D) \arrow[dd, "\mathfrak{a}_{A,(BC),D}"] \\
(AB)(CD) \arrow[d, "\mathfrak{a}_{(AB),C,D}"'] & \\
((AB)C)D & (A(BC))D \arrow[l, "\mathfrak{a}_{A,B,C} 1_D"]
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
A+(0+C) \arrow[r, "\mathfrak{a}_{A,0,C}"] \arrow[d, "1_A+\mathfrak{l}_C"'] & (A+0)+C \arrow[d, "\mathfrak{r}_A+1_C"] \\
A+C \arrow[r, equal] & A+C
\end{tikzcd}$$

(3)

$$\begin{tikzcd}
0+0 \arrow[r, equal] \arrow[d, "\mathfrak{l}_0"'] & 0+0 \arrow[d, "\mathfrak{r}_0"] \\
0 \arrow[r, equal] & 0
\end{tikzcd}$$

(4)

$$\begin{tikzcd}
A+B \arrow[r, "\mathfrak{c}_{A,B}"] \arrow[dr, "1_{A+B}"'] & B+A \arrow[d, "\mathfrak{c}_{B,A}"] \\
 & A+B
\end{tikzcd}$$

(5)

$$\begin{tikzcd}
A+0 \arrow[r, "\mathfrak{c}_{A,0}"] \arrow[d, "\mathfrak{r}_A"'] & 0+A \arrow[d, "\mathfrak{l}_A"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

(6)

$$\begin{tikzcd}
A(BC) \arrow[r, "1_A \mathfrak{c}_{B,C}"] \arrow[d, "\mathfrak{a}_{A,B,C}"'] & A(CB) \arrow[d, "\mathfrak{a}_{A,C,B}"] \\
(AB)C \arrow[d, "\mathfrak{c}_{(AB),C}"'] & (AC)B \arrow[d, "\mathfrak{c}_{A,C} 1_B"] \\
C(AB) \arrow[r, "\mathfrak{a}_{C,A,B}"'] & (CA)B
\end{tikzcd}$$

(Actually Diagram (3) *follows from the other five as shown in* [7]*).*

- *The P-images of $\mathfrak{a}$, $\mathfrak{c}$, $\mathfrak{r}$, and $\mathfrak{l}$, which we denote by the same symbols, are pseudo natural transformations on $\mathcal{C}$ with natural coherence iso 2-cells $\tau^{\mathfrak{a}}$, $\tau^{\mathfrak{c}}$, $\tau^{\mathfrak{l}}$, and $\tau^{\mathfrak{r}}$.*
- *The natural coherence iso 2-cells $\tau^{\mathfrak{a}}, \tau^{\mathfrak{c}}, \tau^{\mathfrak{l}}$, and $\tau^{\mathfrak{r}}$ are coherent with each other in the sense that they satisfy coherence diagrams of the same shape as the coherence diagrams of a symmetric monoidal category. This means that the following diagrams of 2-cells commute, where the unspecified 2-cells are simply identity 2-cells.*

(1) *For*

$$e : A_1 \longrightarrow A_2$$
$$f : B_1 \longrightarrow B_2$$
$$g : C_1 \longrightarrow C_2$$
$$h : D_1 \longrightarrow D_2\,,$$

the 2-cells in the cube

$$
\begin{tikzcd}
A_1(B_1(C_1D_1)) \arrow[rr, "1_{A_1}\mathfrak{a}_{B_1,C_1,D_1}"] \arrow[dr, "e(f(gh))"] \arrow[dd, "\mathfrak{a}_{A_1,B_1,(C_1D_1)}"'] & & A_1((B_1C_1)D_1) \arrow[dr, "e((fg)h)"] \arrow[dddd, "\mathfrak{a}_{A_1,(B_1C_1),D_1}" near start] & \\
& A_2(B_2(C_2D_2)) \arrow[rr, "1_{A_2}\mathfrak{a}_{B_2,C_2,D_2}" near end] \arrow[dd, "\mathfrak{a}_{A_2,B_2,(C_2D_2)}"' near start] & & A_2((B_2C_2)D_2) \arrow[dddd, "\mathfrak{a}_{A_2,(B_2C_2),D_2}"] \\
(A_1B_1)(C_1D_1) \arrow[dr, "(ef)(gh)"'] \arrow[dd, "\mathfrak{a}_{(A_1B_1),C_1,D_1}"'] & & & \\
& (A_2B_2)(C_2D_2) \arrow[dd, "\mathfrak{a}_{(A_2B_2),C_2,D_2}" near end] & & \\
((A_1B_1)C_1)D_1 \arrow[dr, "((ef)g)h"'] & & (A_1(B_1C_1))D_1 \arrow[ll, "\mathfrak{a}_{A_1,B_1,C_1}1_{D_1}" near start] \arrow[dr, "(e(fg))h"] & \\
& ((A_2B_2)C_2)D_2 & & (A_2(B_2C_2))D_2 \arrow[ll, "\mathfrak{a}_{A_2,B_2,C_2}1_{D_2}"]
\end{tikzcd}
$$

commute, where the top, bottom, left, and right faces are inscribed with τ's.

(2) *For $f : A_1 \longrightarrow A_2$ and $g : C_1 \longrightarrow C_2$, the 2-cells in the cube*

$$
\begin{tikzcd}
A_1+(0+C_1) \arrow[rr, "\mathfrak{a}_{A_1,0,C_1}"] \arrow[dr, "f+(1_0+g)"] \arrow[dd, "1_{A_1}+\mathfrak{l}_{C_1}"'] & & (A_1+0)+C_1 \arrow[dr, "(f+1_0)+g"] \arrow[dd, "\mathfrak{r}_{A_1}+1_{C_1}" near end] & \\
& A_2+(0+C_2) \arrow[rr, "\mathfrak{a}_{A_2,0,C_2}" near end] \arrow[dd, "1_{A_2}+\mathfrak{l}_{C_2}" near start] & & (A_2+0)+C_2 \arrow[dd, "\mathfrak{r}_{A_2}+1_{C_2}"] \\
A_1+C_1 \arrow[rr, equal] \arrow[dr, "f+g"'] & & A_1+C_1 \arrow[dr, "f+g"'] & \\
& A_2+C_2 \arrow[rr, equal] & & A_2+C_2
\end{tikzcd}
$$

commute, where the top, left, and right faces are inscribed with τ's.

(3) *For* $1_0 : 0 \longrightarrow 0$*, the 2-cells in the cube*

$$\begin{tikzcd}
0+0 \arrow[rr, equal] \arrow[dr, "1_0+1_0"] \arrow[dd, "\mathfrak{l}_0"'] & & 0+0 \arrow[dr, "1_0+1_0"] \arrow[dd, "\mathfrak{r}_0" near start] & \\
& 0+0 \arrow[rr, equal, crossing over] & & 0+0 \arrow[dd, "\mathfrak{r}_0"] \\
0 \arrow[rr, equal] \arrow[dr, "1_0"'] & & 0 \arrow[dr, "1_0"'] & \\
& 0 \arrow[rr, equal] \arrow[from=uu, "\mathfrak{l}_0" near start, crossing over] & & 0
\end{tikzcd}$$

commute, where the left and right faces are inscribed with τ*'s.*

(4) *For* $f : A_1 \longrightarrow A_2$ *and* $g : B_1 \longrightarrow B_2$*, the 2-cells in the cube*

$$\begin{tikzcd}
A_1+B_1 \arrow[rr, "\mathfrak{c}_{A_1,B_1}"] \arrow[dr, "f+g"] \arrow[dd, "1_{A_1+B_1}"'] & & B_1+A_1 \arrow[dr, "g+f"] \arrow[dd, "\mathfrak{c}_{B_1,A_1}" near start] & \\
& A_2+B_2 \arrow[rr, "\mathfrak{c}_{A_2,B_2}" near start, crossing over] & & B_2+A_2 \arrow[dd, "\mathfrak{c}_{B_2,A_2}"] \\
A_1+B_1 \arrow[rr, "1_{A_1+B_1}" near end] \arrow[dr, "f+g"'] & & A_1+B_1 \arrow[dr, "f+g"'] & \\
& A_2+B_2 \arrow[rr, "1_{A_2+B_2}"'] \arrow[from=uu, "1_{A_2+B_2}"' near start, crossing over] & & A_2+B_2
\end{tikzcd}$$

commute, where the top and right faces are inscribed with τ*'s.*

(5) *For* $f : A_1 \longrightarrow A_2$*, the 2-cells in the cube*

$$\begin{tikzcd}
A_1+0 \arrow[rr, "\mathfrak{c}_{A_1,0}"] \arrow[dr, "f+1_0"] \arrow[dd, "\mathfrak{r}_{A_1}"'] & & 0+A_1 \arrow[dr, "1_0+f"] \arrow[dd, "\mathfrak{l}_{A_1}" near start] & \\
& A_2+0 \arrow[rr, "\mathfrak{c}_{A_2,0}" near start, crossing over] & & 0+A_2 \arrow[dd, "\mathfrak{l}_{A_2}"] \\
A_1 \arrow[rr, equal] \arrow[dr, "f"'] & & A_1 \arrow[dr, "f"'] & \\
& A_2 \arrow[rr, equal] \arrow[from=uu, "\mathfrak{r}_{A_2}"' near start, crossing over] & & A_2
\end{tikzcd}$$

commute, where the top, left, and right faces are inscribed with τ*'s.*

(6) *For*

$$f : A_1 \longrightarrow A_2$$
$$g : B_1 \longrightarrow B_2$$
$$h : C_1 \longrightarrow C_2,$$

the 2-cells in the cube

$$\begin{tikzcd}
A_1(B_1C_1) \arrow[rr, "1_{A_1}\mathfrak{c}_{B_1,C_1}"] \arrow[dr, "f(gh)"] \arrow[dd, "\mathfrak{a}_{A_1,B_1,C_1}"'] & & A_1(C_1B_1) \arrow[dr, "f(hg)"] \arrow[dd, "\mathfrak{a}_{A_1,C_1,B_1}" near start] & \\
& A_2(B_2C_2) \arrow[rr, "1_{A_2}\mathfrak{c}_{B_2,C_2}" near start] \arrow[dd, "\mathfrak{a}_{A_2,B_2,C_2}"' near start] & & A_2(C_2B_2) \arrow[dd, "\mathfrak{a}_{A_2,C_2,B_2}"] \\
(A_1B_1)C_1 \arrow[dr, "(fg)h"'] \arrow[dd, "\mathfrak{c}_{(A_1B_1),C_1}"'] & & (A_1C_1)B_1 \arrow[dr, "(fh)g"] \arrow[dd, "\mathfrak{c}_{A_1,C_1}1_{B_1}" near start] & \\
& (A_2B_2)C_2 \arrow[dd, "\mathfrak{c}_{(A_2B_2),C_2}"' near start] & & (A_2C_2)B_2 \arrow[dd, "\mathfrak{c}_{A_2,C_2}1_{B_2}"] \\
C_1(A_1B_1) \arrow[rr, "\mathfrak{a}_{C_1,A_1,B_1}"' near start] \arrow[dr, "h(fg)"'] & & (C_1A_1)B_1 \arrow[dr, "(hf)g"] & \\
& C_2(A_2B_2) \arrow[rr, "\mathfrak{a}_{C_2,A_2,B_2}"'] & & (C_2A_2)B_2
\end{tikzcd}$$

commute, where the top, bottom, left, and right faces are inscribed with τ's.

- *The bicategory coherence isomorphisms*

$$\alpha^{\mathcal{C}}_{h,g,f} : h(gf) \Longrightarrow (hg)f$$

$$\lambda^{\mathcal{C}}_f : 1_B f \Longrightarrow f \qquad \rho^{\mathcal{C}}_f : f1_A \Longrightarrow f$$

satisfy the usual coherence diagrams for bicategories as follows.

(1) *For composable e, f, g, and h, the associativity pentagon*

$$\begin{tikzcd}
h(g(fe)) \arrow[r, Rightarrow, "i_h*\alpha_{g,f,e}"] \arrow[d, Rightarrow, "\alpha_{h,g,(fe)}"'] & h((gf)e) \arrow[dd, Rightarrow, "\alpha_{h,(gf),e}"] \\
(hg)(fe) \arrow[d, Rightarrow, "\alpha_{(hg),f,e}"'] & \\
((hg)f)e & (h(gf))e \arrow[l, Rightarrow, "\alpha_{h,g,f}*i_e"]
\end{tikzcd}$$

commutes.

(2) *For* $A \xrightarrow{f} B \xrightarrow{g} C$, *the left and right identity coherence 2-cells commute.*

$$\begin{tikzcd} g(1_B f) \arrow[r, Rightarrow, "\alpha^{\mathcal{C}}_{g,1_B,f}"] \arrow[d, Rightarrow, "i_g * \lambda^{\mathcal{C}}_f"'] & (g1_B)f \arrow[d, Rightarrow, "\rho^{\mathcal{C}}_g * i_f"] \\ gf \arrow[r, equal] & gf \end{tikzcd}$$

- *There are coherence iso 2-cells*

$$m_{A+B} \xRightarrow{\epsilon_{A,B}} m_A + m_B \qquad n_{A+B} \xRightarrow{\zeta_{A,B}} n_A + n_B$$

$$(1_A + n_A) \circ (m_A + 1_A) \xRightarrow{\eta_A} 1_A$$

$$(n_A + 1_A) \circ (1_A + m_A) \xRightarrow{\theta_A} 1_A$$

which satisfy the coherence diagrams below.

$$\begin{tikzcd} (1_{A+B} + n_{A+B}) \circ (m_{A+B} + 1_{A+B}) \arrow[r, Rightarrow, "\eta_{A+B}"] \arrow[d, Rightarrow] & 1_{A+B} \arrow[d, Rightarrow] \\ (1_A + n_A) \circ (m_A + 1_A) + (1_B + n_B) \circ (m_B + 1_B) \arrow[r, Rightarrow, "\eta_A + \eta_B"'] & 1_A + 1_B \end{tikzcd}$$

$$\begin{tikzcd} (n_{A+B} + 1_{A+B}) \circ (1_{A+B} + m_{A+B}) \arrow[r, Rightarrow, "\theta_{A+B}"] \arrow[d, Rightarrow] & 1_{A+B} \arrow[d, Rightarrow] \\ (n_A + 1_A) \circ (1_A + m_A) + (n_B + 1_B) \circ (1_B + m_B) \arrow[r, Rightarrow, "\theta_A + \theta_B"'] & 1_A + 1_B \end{tikzcd}$$

Note that these coherence iso 2-cells $\epsilon, \zeta, \eta, \theta$ and coherence diagrams of 2-cells actually involve the (images of the) coherence isos of the underlying symmetric monoidal category I. For example, ϵ is actually a 2-cell of the form

$$\begin{tikzcd} 0 \arrow[r, "m_{A+B}"] \arrow[d, "\cong"'] & (A+B)+(A+B) \arrow[d, "\cong"] \\ 0+0 \arrow[r, "m_A + m_B"'] \arrow[ur, Rightarrow, "\epsilon_{A,B}"] & (A+A)+(B+B) \end{tikzcd}$$

where the vertical isos come from the coherence isos of I.

Proof. This theorem is a special case of Theorem 7.9. $\square$

Remark 3.8. If we leave off the assumption that I is a groupoid in the theorem, then there are subtleties with covariance and contravariance. If we leave off the assumption on strict identity morphisms, then the map $I \longrightarrow \mathcal{C}$ is no longer a strict 2-functor, instead it is a pseudo functor.

4. Worldsheets revisited

In this section we will give the technical details of the first, more naive approach to the bicategory of worldsheets, which will make it into a pseudo Frobenius symmetric monoidal category. Recall that we need to define a symmetric monoidal category I, a strict 2-functor $X : I^2 \longrightarrow Cat$, and operations $+, 0, \circ, 1_A, m_A, n_A$ as well as coherence isomorphisms that satisfy coherence diagrams.

The symmetric monoidal category I has finite sets as its objects and bijections as its morphisms. The operation $+$ is given by the disjoint union

$$A \coprod B := A \times \{1\} \cup B \times \{2\}$$

which has the empty set $\emptyset$ as its unit 0. The associativity coherence iso is

$$A \coprod (B \coprod C) \longrightarrow (A \coprod B) \coprod C$$

$$(a, 1) \longmapsto (a, 1, 1)$$

$$(b, 1, 2) \longmapsto (b, 2, 1)$$

$$(c, 2, 2) \longmapsto (c, 2)$$

while the left and right unit coherence isos are

$$0 \coprod A \longrightarrow A$$

$$(a, 2) \longmapsto a$$

$$A \coprod 0 \longrightarrow A$$

$$(a, 1) \longmapsto a.$$

The symmetry is given by

$$A \coprod B \longrightarrow B \coprod A$$

$$(a, 1) \longmapsto (a, 2)$$

$$(b, 2) \longmapsto (b, 1).$$

It is routine to verify that these coherence isos satisfy the six diagrams for a symmetric monoidal category, which in turn says that I is a pseudo algebra over the theory of commutative monoids.

The next piece of structure we need is a strict 2-functor $X : I^2 \longrightarrow Cat$. An object of $X_{A,B}$ is a topological space x such that:

- Each component of x is a connected worldsheet (real, two dimensional, compact, smooth manifold with analytic boundary parametrizations and a complex structure), or a one dimensional smooth manifold diffeomorphic to S^1, or a space homeomorphic to the point at infinity of the 1-point compactification of the moduli space of elliptic curves over $\mathbb{C}$. This point at infinity is a $\mathbb{P}^1$ with three marked points, two of which are identified. We denote by x^{world} the subspace of x consisting of worldsheet components and we denote by x^{circle} the subspace of x consisting of circle components.
- The space x is equipped with two functions

  $$s^{\text{in}}, s^{\text{out}} : \pi_0(x^{\text{circle}}) \longrightarrow \{\emptyset, \{1\}, \{1, 2\}\}$$

  which satisfy

  $$|s^{\text{in}}(\alpha)| + |s^{\text{out}}(\alpha)| = 2$$

  for each $\alpha \in \pi_0(x^{\text{circle}})$. We say that α has $|s^{\text{in}}(\alpha)|$ *inbound components and* $|s^{\text{out}}(\alpha)|$ *outbound components*. We conceive of α as an infinitely thin annulus.
- The space x is equipped with a bijection between the set A and the set

  $$\{\text{inbound } \partial \text{ components of } x^{\text{world}}\} \quad \bigcup \quad \bigcup_{\alpha \in \pi_0(x^{\text{circle}})} s^{\text{in}}(\alpha) \times \alpha.$$

- The space x is equipped with a bijection between the set B and the set

$$\{\text{outbound } \partial \text{ components of } x^{\text{world}}\} \bigcup \bigcup_{\alpha \in \pi_0(x^{\text{circle}})} s^{\text{out}}(\alpha) \times \alpha.$$

A morphism $x \longrightarrow y$ in $X_{A,B}$ is a homeomorphism which takes x^{world} to y^{world} smoothly, takes x^{circle} to y^{circle} smoothly, preserves the bijections with A and B and also preserves the boundary parametrizations.

We next describe the various operations on the 2-functor X. The operation $+ : X_{A,B} \times X_{C,D} \longrightarrow X_{A+C,B+D}$ is given by disjoint union

$$x + y := x \coprod y = x \times \{1\} \cup y \times \{2\}.$$

The object $0 \in X_{0,0}$ is simply the empty set.

The operation

$$\begin{aligned} X_{B,C} \times X_{A,B} &\xrightarrow{\circ} X_{A,C} \\ (y, x) &\longmapsto y \circ x \end{aligned}$$

is obtained by first taking the disjoint union $y \coprod x$, then identifying (gluing) any two circles with the 'same' label in the following way. There are several cases.

(1) Gluing an outbound boundary component of x^{world} to an inbound boundary component of y^{world},
(2) Gluing a circle in x^{circle} which has one inbound label and one outbound label to an inbound boundary component of y^{world},
(3) Gluing a circle in x^{circle} which has one inbound label and one outbound label to a circle in y^{circle} which has at least one inbound label,
(4) Gluing a circle x^{circle} which has two outbound labels to two inbound boundary components of y^{world},
(5) Gluing a circle in x^{circle} which has two outbound labels to two circles in y^{circle}, each of which has at least one inbound label,
(6) Gluing a circle in x^{circle} which has two outbound labels to a circle in y^{circle} which has two inbound labels,
(7) Gluing a circle in y^{circle} to parts of x.

The gluing procedures go as follows.

(1) If a boundary component of x^{world} has outbound label $(b, 2)$ and a boundary component of y^{world} has inbound label $(b, 1)$, then they are glued according to

$$f^{(b,1)}(z) \sim f^{(b,2)}(z)$$

for all $z \in S^1$.
(2) If $\alpha \in \pi_0(x^{\text{circle}})$ has inbound label $(a, 2)$ and outbound label $(b, 2)$, and a boundary component of y^{world} has inbound label $(b, 1)$, then we keep the parametrization of that inbound boundary component in $y \coprod x$, but give it the inbound label $(a, 2)$. We then remove α from $y \coprod x$ (we conceive of α as being glued onto this boundary component of y^{world}).
(3) If $\alpha \in \pi_0(x^{\text{circle}})$ has an inbound label $(a, 2)$ and an outbound label $(b, 2)$, and $\beta \in \pi_0(y^{\text{circle}})$ has an inbound label $(b, 1)$, then β gets an inbound label $(a, 2)$ instead of $(b, 1)$. The other label on β stays the same, regardless of whether it is inbound or outbound. We then remove α from $y \coprod x$ (we conceive of the circle α as being glued onto the circle β).
(4) If $\alpha \in \pi_0(x^{\text{circle}})$ has two outbound labels $(b, 2)$ and $(b', 2)$ while y^{world} has two inbound boundary components with labels $(b, 1)$ and $(b', 1)$ respectively, then those two boundary components are glued according to

$$f^{(b,1)}(z) \sim f^{(b',1)}(\phi(z))$$

for all $z \in S^1$, where $\phi : [0, 1] \longrightarrow S^1$ is the map $t \mapsto \exp(-2\pi i t)$. We then remove α from $y \coprod x$ (we conceive of α as being glued between the two boundary components).

(5) If $\alpha \in \pi_0(x^{\text{circle}})$ has two outbound labels $(b, 2)$ and $(b', 2)$ and there are circles $\beta, \beta' \in \pi_0(y^{\text{circle}})$ with inbound labels $(b, 1)$ and $(b', 1)$ and other labels $(c, 1)$ and $(c', 1)$, then β and β' are fused together to give a new circle with two labels $(c, 1)$ and $(c', 1)$ (with the same orientation as they had in β and β'). We then remove α, β, and β' from $y \coprod x$ (we conceive of α as being glued between the two circles β and β' to give a new circle comprised of all three).
(6) If $\alpha \in \pi_0(x^{\text{circle}})$ has two outbound labels and $\beta \in \pi_0(y^{\text{circle}})$ also has the same two inbound labels, then α and β fuse together to give the point at infinity of the 1-point compactification of the moduli space of elliptic curves over $\mathbb{C}$. We then remove α and β from $y \coprod x$ (we conceive of α and β as being glued together).
(7) Gluing a circle from $\pi_0(y^{\text{circle}})$ to parts of x is similar to (2)–(6) above.

After this gluing we replace the labels $(a, 2)$ by a and $(c, 1)$ by c and obtain an element $y \circ x$ of $X_{A,C}$. This concludes the definition of the operation $\circ$.

We still need to define the operations 0, 1_A, m_A, and n_A. The object

$$1_A \in X_{A,A}$$

is $S^1 \times A$ where $S^1 \times \{a\}$ has inbound label a and outbound label a. The object

$$m_A \in X_{0,A+A}$$

is $S^1 \times A$ where $S^1 \times \{a\}$ has two outbound labels $(a, 1)$ and $(a, 2)$. The object

$$n_A \in X_{A+A,0}$$

is $S^1 \times A$ where $S^1 \times \{a\}$ has two inbound labels $(a, 1)$ and $(a, 2)$.

The question of coherence can be treated as in diagrams (11)–(13) of [5]. We have just proved the following theorem.

Theorem 4.1. *The worldsheets as described above form a pseudo Frobenius symmetric monoidal category as defined in Section 3.*

Remark 4.2. The approach we followed here, although it produces a pseudo Frobenius symmetric monoidal category, has an aesthetic flaw (which, as is often the case, ultimately turns out to be a material flaw): it requires the 1-point compactification of the moduli space of elliptic curves. The 1-point compactification is not necessary when the degenerate annuli are excluded. One could alternately work with worldsheets without allowing the degenerate annuli, as we do in the following section using the cobordism 2-theory.

5. Cobordism implies commutative monoid with cancellation

As we saw in Section 4, one must artificially include infinitely thin annuli in the class of worldsheets and undertake other alterations to make the class of worldsheets into a pseudo algebra over the 2-theory of Frobenius symmetric monoidal categories. This however is not entirely satisfactory for the cobordism definition of conformal field theory, since the images of the infinitely thin annuli under a functor into the category of Hilbert spaces will not be Hilbert–Schmidt. However, conformal field theories require the images of cobordisms to be Hilbert–Schmidt operators. In this section we propose pseudo algebras over the *cobordism 2-theory* as the appropriate formalism for the cobordism definition of conformal field theory. No artificial changes are required to make the class of worldsheets into a pseudo algebra over this 2-theory.

Definition 5.1. A *strict algebra over the cobordism 2-theory* consists of a commutative monoid $(I, +)$ and a function $X : I^2 \longrightarrow Sets$ with natural operations

(1) *Disjoint union*

$$X_{A,B} \times X_{C,D} \xrightarrow{+} X_{A+C,B+D}.$$

(2) *Unit*

$$\{*\} \xrightarrow{0} X_{0,0}.$$

(3) *Composition*

$$X_{B,C} \times X_{A,B} \xrightarrow{\circ} X_{A,C}.$$

(4) For each $n \geq 1$, and $1 \leq j \leq n$, and $A_1, \ldots, A_n \in I$, we have an operation of *slicing annuli from the source*

$$X_{A_1+A_2+\cdots+A_n,B} \xrightarrow{\phi_j^n} X_{A_j,A_j}.$$

(5) For each $n \geq 1$, and $1 \leq j \leq n$, and $B_1, \ldots, B_n \in I$, we have an operation of *slicing annuli from the target*

$$X_{A,B_1+B_2+\cdots+B_n} \xrightarrow{\psi_j^n} X_{B_j,B_j}.$$

(6) The *remainder* after slicing annuli from source and target

$$X_{A,B} \xrightarrow{\xi} X_{A,B}.$$

(7) *Weak Frobenius elements* for all $A \in I$ and $i = 1, 2$

$$X_{A,A} \xrightarrow{m_A^i} X_{0,A+A}$$

$$X_{A,A} \xrightarrow{n_A^i} X_{A+A,0}.$$

(8) *Factorising annuli* for all $A \in I$ and $i = 1, 2$

$$X_{A,A} \xrightarrow{p_A^i} X_{A,A}$$

$$X_{A,A} \xrightarrow{q_A^i} X_{A,A}.$$

These natural operations are required to satisfy the following *axioms*.

(1) Composition is associative.

(2) 0 is a unit for $+$.

(3) For $x \in X_{B,0}$ and $y \in X_{0,B}$ we have

$$0 \circ x = x \qquad\qquad y \circ 0 = y.$$

(4) The composition $X_{0,C} \times X_{A,0} \xrightarrow{\circ} X_{A,C}$ is the same as disjoint union $X_{0,C} \times X_{A,0} \xrightarrow{+} X_{A,C}$.

(5) If $A_j = 0$ in (4), then $\phi_j^n(x) = 0$.

(6) If $B_j = 0$ in (5), then $\psi_j^n(x) = 0$.

(7) Slicing annuli is compatible with $+$, in the sense that

$$\begin{tikzcd}
X_{A_1+A_2+\cdots+A_n,B} \arrow[r, "\phi_1^n\times\cdots\times\phi_n^n"] \arrow[dr, "\phi_1^1"'] & X_{A_1,A_1}\times\cdots\times X_{A_n,A_n} \arrow[d, "+\cdots+"] \\
 & X_{A_1+A_2+\cdots+A_n,A_1+A_2+\cdots+A_n}
\end{tikzcd}$$

and

$$\begin{tikzcd}
X_{A,B_1+B_2+\cdots+B_n} \arrow[r, "\psi_1^n\times\cdots\times\psi_n^n"] \arrow[dr, "\phi_1^1"'] & X_{B_1,B_1}\times\cdots\times X_{B_n,B_n} \arrow[d, "+\cdots+"] \\
 & X_{B_1+B_2+\cdots+B_n,B_1+B_2+\cdots+B_n}
\end{tikzcd}$$

commute.

(8) The remainder after slicing annuli is compatible with $+$ in the sense that

$$\begin{tikzcd}
X_{A,B}\times X_{C,D} \arrow[r, "+"] \arrow[d, "\xi\times\xi"'] & X_{A+C,B+D} \arrow[d, "\xi"] \\
X_{A,B}\times X_{C,D} \arrow[r, "+"'] & X_{A+C,B+D}
\end{tikzcd}$$

commutes.

(9) Slicing annuli from the source and target and gluing back on gives the same surface in the sense that

$$\begin{tikzcd}
X_{A,B} \arrow[r, "\psi_1^1\times\xi\times\phi_1^1"] \arrow[dr, "1_{X_{A,B}}"'] & X_{B,B}\times X_{A,B}\times X_{A,A} \arrow[d, "\circ(\circ)"] \\
 & X_{A,B}
\end{tikzcd}$$

commutes.

(10) The weak Frobenius elements are compatible with $+$.

$$m^i_{A+B} = m^i_A + m^i_B$$
$$n^i_{A+B} = n^i_A + n^i_B.$$

(11) The factorising annuli are compatible with $+$.

$$p^i_{A+B} = p^i_A + p^i_B$$
$$q^i_{A+B} = q^i_A + q^i_B.$$

(12) The sliced annuli factor in a Frobenius way, in the sense that the following diagrams commute.

$$\begin{tikzcd}
X_{A,B} \arrow[r, "\phi_1^1"] & X_{A,A} \arrow[r, "q_A^1 \times n_A^1 \times m_A^1 \times p_A^1"] \arrow[dr, "1_{X_{A,A}}"'] & X_{A,A} \times X_{A+A,0} \times X_{0,A+A} \times X_{A,A} \arrow[d, "\circ(+\times+)"] \\
& & X_{A,A}
\end{tikzcd}$$

$$\begin{tikzcd}
X_{A,B} \arrow[r, "\phi_1^1"] & X_{A,A} \arrow[r, "n_A^2 \times q_A^2 \times p_A^2 \times m_A^2"] \arrow[dr, "1_{X_{A,A}}"'] & X_{A+A,0} \times X_{A,A} \times X_{A,A} \times X_{0,A+A} \arrow[d, "\circ(+\times+)"] \\
& & X_{A,A}
\end{tikzcd}$$

$$\begin{tikzcd}
X_{A,B} \arrow[r, "\psi_1^1"] & X_{B,B} \arrow[r, "q_B^1 \times n_B^1 \times m_B^1 \times p_B^1"] \arrow[dr, "1_{X_{B,B}}"'] & X_{B,B} \times X_{B+B,0} \times X_{0,B+B} \times X_{B,B} \arrow[d, "\circ(+\times+)"] \\
& & X_{B,B}
\end{tikzcd}$$

$$\begin{tikzcd}
X_{A,B} \arrow[r, "\psi_1^1"] & X_{B,B} \arrow[r, "n_B^2 \times q_B^2 \times p_B^2 \times m_B^2"] \arrow[dr, "1_{X_{B,B}}"'] & X_{B+B,0} \times X_{B,B} \times X_{B,B} \times X_{0,B+B} \arrow[d, "\circ(+\times+)"] \\
& & X_{B,B}
\end{tikzcd}$$

Example 5.2. The worldsheets (without infinitely thin annuli) form a *pseudo algebra over the cobordism 2-theory*. Here I is the symmetric monoidal category of finite sets and bijections under the operation of disjoint union. The objects of the category $X_{A,B}$ are (not necessarily connected) worldsheets (real, two dimensional, compact, smooth manifolds with analytic boundary parametrizations and a complex structure) equipped with a bijection between the inbound boundary components and A as well as a bijection between the outbound boundary components and B. The morphisms of $X_{A,B}$ are holomorphic diffeomorphisms which preserve the bijections and the boundary parametrizations. The operation $+$ is disjoint union of worldsheets, the unit 0 is the empty manifold, and the composition is the gluing of surfaces according to the boundary labellings. The remaining axioms say that each worldsheet is equipped with a collaring, and that each cylinder is equipped with two Frobenius-type decompositions, and these collarings and decompositions are compatible with disjoint union. The term 'annuli' is to be interpreted loosely, because the operations ϕ_1^1 and ψ_1^1 applied to a worldsheet with several boundary components slice off a union of annuli rather than a single annulus.

Theorem 5.3. *A strict algebra* $X : I^2 \longrightarrow$*Sets over the cobordism 2-theory gives rise to a strict commutative monoid with cancellation with the same underlying function* $X : I^2 \longrightarrow$*Sets, the same commutative monoid* I*, the same operation* $+$*, and the same unit* $0 \in X_{0,0}$.

Proof. We only need to define the cancellation

$$\check{?} : X_{A+C,B+C} \longrightarrow X_{A,B}$$

in terms of the operations of the algebra over the cobordism 2-theory. It suffices to construct $\check{?} : X_{C,C} \longrightarrow X_{0,0}$. The picture to keep in mind is the self gluing of a cylinder to get a torus: first cut the cylinder into three, then factor the last piece in a Frobenius way, then cut one of the Frobenius elements into an elbow and two cylinders, then appropriately glue five of the cylinders together, and put the resulting cylinder and the other remaining cylinder between the two elbows to obtain a torus.

Formally for $x \in X_{C,C}$, we first slice off annuli as

$$x = \psi_1^1(x) \circ \xi(x) \circ \phi_1^1(x).$$

The second annulus factors in a Frobenius way as

$$\psi_1^1(x) = (n_C^2(\psi_1^1(x)) + q_C^2(\psi_1^1(x))) \circ (p_C^2(\psi_1^1(x)) + m_C^2(\psi_1^1(x))).$$

The Frobenius element $m_C^2(\psi_1^1(x))$ factors into an elbow and two annuli as

$$\begin{aligned} m_C^2(\psi_1^1(x)) &= 0 \circ \xi(m_C^2(\psi_1^1(x))) \circ (\psi_1^2(m_C^2(\psi_1^1(x))) + \psi_2^2(m_C^2(\psi_1^1(x)))) \\ &= \xi(m_C^2(\psi_1^1(x))) \circ (\psi_1^2(m_C^2(\psi_1^1(x))) + \psi_2^2(m_C^2(\psi_1^1(x)))). \end{aligned}$$

Then we define $\check{x}$ as

$$\begin{aligned} & n_C^2(\psi_1^1(x)) \circ ((p_C^2(\psi_1^1(x)) \circ \xi(x) \circ \phi_1^1(x) \circ q_C^2(\psi_1^1(x)) \\ & \circ \psi_2^2(m_C^2(\psi_1^1(x)))) + \psi_1^2(m_C^2(\psi_1^1(x)))) \circ \xi(m_C^2(\psi_1^1(x))). \quad \square \end{aligned}$$

Theorem 5.4. *There is a natural morphism of 2-theories over the theory of commutative monoids from the 2-theory of commutative monoids with cancellation to the cobordism 2-theory.*

Proof. The word $+$ in the 2-theory of commutative monoids with cancellation maps to the word $+$ in the cobordism 2-theory, the image of the cancellation word $\check{?}$ is defined as in Theorem 5.3. $\square$

A version of Theorem 3.7 also holds, but we need two definitions before its statement.

Definition 5.5. A *bicategory without units* $\mathcal{C}$ is comprised of a collection Obj $\mathcal{C}$ of *objects* $A, B, C \ldots$, categories $\mathrm{Mor}_{\mathcal{C}}(A, B) = \mathcal{C}(A, B)$, functors

$$\mathrm{Mor}_{\mathcal{C}}(B, C) \times \mathrm{Mor}_{\mathcal{C}}(A, B) \xrightarrow{\circ} \mathrm{Mor}_{\mathcal{C}}(A, C),$$

and natural isomorphisms called *coherence isomorphisms*

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{C}}(C, D) \times \mathrm{Mor}_{\mathcal{C}}(B, C) \times \mathrm{Mor}_{\mathcal{C}}(A, B) \arrow[r, "\circ \times 1_{\mathrm{Mor}_{\mathcal{C}}(A,B)}"] \arrow[d, "1_{\mathrm{Mor}_{\mathcal{C}}(C,D)} \times \circ"'] & \mathrm{Mor}_{\mathcal{C}}(B, D) \times \mathrm{Mor}_{\mathcal{C}}(A, B) \arrow[d, "\circ"] \\ \mathrm{Mor}_{\mathcal{C}}(C, D) \times \mathrm{Mor}_{\mathcal{C}}(A, C) \arrow[r, "\circ"'] \arrow[ur, Rightarrow, "\alpha_{A,B,C}"] & \mathrm{Mor}_{\mathcal{C}}(A, D), \end{tikzcd}$$

such that the following *coherence diagram* is satisfied. For composable e, f, g, and h the associativity pentagon

$$\begin{tikzcd}
h(g(fe)) \arrow[r, Rightarrow, "i_h * \alpha_{g,f,e}"] \arrow[d, Rightarrow, "\alpha_{h,g,(fe)}"'] & h((gf)e) \arrow[dd, Rightarrow, "\alpha_{h,(gf),e}"] \\
(hg)(fe) \arrow[d, Rightarrow, "\alpha_{(hg),f,e}"'] & \\
((hg)f)e & (h(gf))e \arrow[l, Rightarrow, "\alpha_{h,g,f} * i_e"]
\end{tikzcd}$$

commutes.

Definition 5.6. A *homomorphism of bicategories without units* $G : \mathcal{C} \longrightarrow \mathcal{C}'$ consists of a function

$$G : \text{Obj } \mathcal{C} \longrightarrow \text{Obj } \mathcal{C}',$$

functors

$$G_{A,B} : \text{Mor}_{\mathcal{C}}(A, B) \longrightarrow \text{Mor}_{\mathcal{C}'}(GA, GB),$$

and natural isomorphisms (*coherence isomorphisms*)

$$\begin{tikzcd}
\text{Mor}_{\mathcal{C}}(B, C) \times \text{Mor}_{\mathcal{C}}(A, B) \arrow[r, "\circ"] \arrow[d, "G_{B,C} \times G_{A,B}"'] & \text{Mor}_{\mathcal{C}}(A, C) \arrow[d, "G_{A,C}"] \\
\text{Mor}_{\mathcal{C}'}(GB, GC) \times \text{Mor}_{\mathcal{C}'}(GA, GB) \arrow[r, "\circ"'] \arrow[ur, Rightarrow, "\gamma_{A,B,C}"] & \text{Mor}_{\mathcal{C}'}(GA, GC),
\end{tikzcd}$$

such that the *coherence diagram*

$$\begin{tikzcd}
Gh \circ (Gf \circ Gf) \arrow[r, Rightarrow, "i_{Gh} * \gamma_{g,f}"] \arrow[d, Rightarrow, "\alpha^{\mathcal{C}'}_{Gh,Gg,Gf}"'] & Gh \circ G(g \circ f) \arrow[r, Rightarrow, "\gamma_{h,g \circ f}"] & G(h \circ (g \circ f)) \arrow[d, Rightarrow, "G(\alpha^{\mathcal{C}}_{h,g,f})"] \\
(Gh \circ Gg) \circ Gf \arrow[r, Rightarrow, "\gamma_{h,g} * i_{Gf}"'] & G(h \circ g) \circ Gf \arrow[r, Rightarrow, "\gamma_{h \circ g, f}"'] & G((h \circ g) \circ f)
\end{tikzcd}$$

commutes.

Theorem 5.7. *The 2-category of pseudo algebras over the cobordism 2-theory with underlying groupoid I is biequivalent to the 2-category of bicategories $\mathcal{C}$ without identities, equipped with an operation*

$$+ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C},$$

an object $0 \in \mathcal{C}$, and functors

$$\mathcal{C}(A_1 + A_2 + \cdots + A_n, B) \xrightarrow{\phi_j^n} \mathcal{C}(A_j, A_j)$$

$$\mathcal{C}(A, B_1 + B_2 + \cdots + B_n) \xrightarrow{\psi_j^n} \mathcal{C}(B_j, B_j)$$

$$\mathcal{C}(A, B) \xrightarrow{\xi} \mathcal{C}(A, B)$$

$$\mathcal{C}(A, A) \xrightarrow{m_A^i} \mathcal{C}(0, A + A)$$

$$\mathcal{C}(A, A) \xrightarrow{n_A^i} \mathcal{C}(A + A, 0)$$

$$\mathcal{C}(A, A) \xrightarrow{p_A^i} \mathcal{C}(A, A)$$

$$\mathcal{C}(A, A) \xrightarrow{q_A^i} \mathcal{C}(A, A),$$

for $i = 1, 2$ as well as a groupoid I equipped with $+ : I \times I \longrightarrow I$, such that:

- *$+ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a homomorphism of bicategories without identities.*
- Obj $I =$ Obj $\mathcal{C}$ *and* $A +_{\mathcal{C}} B = A +_I B$.
- *There is a composition $f \circ j$ and $k \circ f$ for morphisms f of $\mathcal{C}$ and morphism j, k of I which has the same properties as a composition defined by $f \circ j = f \circ P(j)$ and $k \circ f = P(k) \circ f$ for a homomorphism $P : I \longrightarrow \mathcal{C}$ of bicategories without identities.*
- *The functors $\phi_j^n, \ldots, q_A^i$ above are natural in the sense of Definition 7.6.*
- *$(I, +)$ is a symmetric monoidal category with coherence isos*

$$\mathfrak{a}_{A,B,C} : A + (B + C) \xrightarrow{\cong} (A + B) + C \qquad \mathfrak{c}_{A,B} : A + B \xrightarrow{\cong} B + A$$

$$\mathfrak{l}_A : 0 + A \xrightarrow{\cong} A \qquad \mathfrak{r}_A : A + 0 \xrightarrow{\cong} A.$$

- *The operation $+$ on $\mathcal{C}$ is associative up to a coherence isomorphism $\mathfrak{a}^{\mathcal{C}}$ which satisfies the pentagon axiom, and composition of a morphism of $\mathcal{C}$ with $\mathfrak{a}^I$ is the same as composition with $\mathfrak{a}^{\mathcal{C}}$. We write $\mathfrak{a}$ for both $\mathfrak{a}^I$ and $\mathfrak{a}^{\mathcal{C}}$.*
- *$\mathfrak{a}^{\mathcal{C}}$ is a pseudo natural transformation on $\mathcal{C}$ with natural iso 2-cell $\tau^{\mathfrak{a}}$.*
- *The natural iso 2-cell $\tau^{\mathfrak{a}}$ is coherent: for*

$$e : A_1 \longrightarrow A_2$$
$$f : B_1 \longrightarrow B_2$$
$$g : C_1 \longrightarrow C_2$$
$$h : D_1 \longrightarrow D_2,$$

the 2-cells in the cube

$$\begin{tikzcd}
A_1(B_1(C_1D_1)) \arrow[rr, "1_{A_1}\mathfrak{a}_{B_1,C_1,D_1}"] \arrow[dr, "e(f(gh))"] \arrow[dd, "\mathfrak{a}_{A_1,B_1,(C_1D_1)}"'] & & A_1((B_1C_1)D_1) \arrow[dr, "e((fg)h)"] \arrow[dddd, "\mathfrak{a}_{A_1,(B_1C_1),D_1}"] & \\
& A_2(B_2(C_2D_2)) \arrow[rr, "1_{A_2}\mathfrak{a}_{B_2,C_2,D_2}"] \arrow[dd, "\mathfrak{a}_{A_2,B_2,(C_2D_2)}"'] & & A_2((B_2C_2)D_2) \arrow[dddd, "\mathfrak{a}_{A_2,(B_2C_2),D_2}"] \\
(A_1B_1)(C_1D_1) \arrow[dr, "(ef)(gh)"'] \arrow[dd, "\mathfrak{a}_{(A_1B_1),C_1,D_1}"'] & & & \\
& (A_2B_2)(C_2D_2) \arrow[dd, "\mathfrak{a}_{(A_2B_2),C_2,D_2}"] & & \\
((A_1B_1)C_1)D_1 \arrow[dr, "((ef)g)h"'] & & (A_1(B_1C_1))D_1 \arrow[ll, "\mathfrak{a}_{A_1,B_1,C_1}1_{D_1}"] \arrow[dr, "(e(fg))h"] & \\
& ((A_2B_2)C_2)D_2 & & (A_2(B_2C_2))D_2 \arrow[ll, "\mathfrak{a}_{A_2,B_2,C_2}1_{D_2}"]
\end{tikzcd}$$

commute, where the top, bottom, left, and right faces are inscribed with τ's.

- *For each equation in axioms* (1)–(12) *of Definition* 5.1 *we have a coherence isomorphism. These are coherent in the sense that any two composites of coherence isomorphisms with the same source and target are equal. For example for* (10)*, there are coherence iso 2-cells*

$$m^i_{A+B}(f+g) \xRightarrow{\epsilon^i_{A,B}(f,g)} m^i_A(f) + m^i_B(g)$$

$$n^i_{A+B}(f+g) \xRightarrow{\zeta^i_{A,B}(f,g)} n^i_A(f) + n^i_B(g)$$

$$(q^1_A(\phi^1_1(f)) + n^1_A(\phi^1_1(f))) \circ (m^1_A(\phi^1_1(f)) + p^1_A(\phi^1_1(f))) \xRightarrow{\eta^\phi_A(f)} \phi^1_1(f)$$

$$(n^2_A(\phi^1_1(f)) + q^2_A(\phi^1_1(f))) \circ (p^2_A(\phi^1_1(f)) + m^2_A(\phi^1_1(f))) \xRightarrow{\theta^\phi_A(f)} \phi^1_1(f)$$

$$(q^1_A(\psi^1_1(f)) + n^1_A(\psi^1_1(f))) \circ (m^1_A(\psi^1_1(f)) + p^1_A(\psi^1_1(f))) \xRightarrow{\eta^\psi_A(f)} \psi^1_1(f)$$

$$(n^2_A(\psi^1_1(f)) + q^2_A(\psi^1_1(f))) \circ (p^2_A(\psi^1_1(f)) + m^2_A(\psi^1_1(f))) \xRightarrow{\theta^\psi_A(f)} \psi^1_1(f)$$

which satisfy four coherence diagrams analogous to the last two coherence diagrams of Theorem 3.7.

Proof. From a pseudo algebra $X : I^2 \longrightarrow Cat$ over the cobordism 2-theory, we obtain a bicategory without units by Obj $\mathcal{C}$:= Obj I and $\mathcal{C}(A, B) := X_{A,B}$. Composition and $+$ in $\mathcal{C}$ are the composition and disjoint union in X. For $f \in \mathcal{C}(B, C) = X_{B,C}$, $j \in I(A, B)$, and $k \in I(C, D)$, the mixed compositions are defined by

$$f \circ j := X_{j^{-1},1_C}(f)$$
$$k \circ f := X_{1_B,k}(f).$$

The functors $\phi, \psi, \xi, m^i, n^i, p^i$, and q^i in $\mathcal{C}$ are the operations in X.

From a bicategory without units $\mathcal{C}$ with structure as in the statement of the theorem, we obtain a pseudo algebra $X : I^2 \longrightarrow Cat$ over the cobordism 2-theory by $X_{A,B} := \mathcal{C}(A, B)$ and $X_{j^{-1},k}(f) := k \circ f \circ j$.

The rest of the proof is analogous to the proofs of Theorems 7.5 and 7.9. □

6. The comparison

A conformal field theory is a morphism of a certain structure. In the commutative monoid with cancellation approach, a conformal field theory is a morphism of pseudo commutative monoids with cancellation, while in the cobordism approach a conformal field theory is a morphism of pseudo algebras over the cobordism 2-theory, where the target is a pseudo algebra that comes from a pseudo commutative monoid with cancellation. Typically the source is comprised of the worldsheets and the target is comprised of Hilbert spaces. The worldsheets were fully described and modified in Section 4 to make them a pseudo Frobenius symmetric monoidal category, but in this section we use the unmodified worldsheets as in Example 5.2. We follow the convention that *morphisms* of pseudo algebras are *pseudo morphisms* of pseudo algebras, *i.e.* operations are preserved up to coherence isomorphisms that satisfy coherence diagrams. After reviewing conformal field theory in the two approaches, we will see that a conformal field theory with modular functor in the cobordism approach gives rise to a conformal field theory with modular functor in the commutative monoid with cancellation approach.

6.1. *The commutative monoid with cancellation approach to conformal field theory*

A pseudo commutative monoid with cancellation is a symmetric monoidal category I and a strict 2-functor $X : I^2 \longrightarrow Cat$ equipped with three natural functors

$$+ : X_{A,B} \times X_{C,D} \longrightarrow X_{A+C,B+D}$$

$$\check{?} : X_{A+C,B+C} \longrightarrow X_{A,B}$$

$$0 \in X_{0,0}$$

called *disjoint union, cancellation,* and *unit* which satisfy the axioms of Example A.17 up to coherence isomorphisms satisfying coherence diagrams. These axioms say that disjoint union $+$ is commutative, associative, and has unit 0. The axioms require further that cancellation $\check{?}$ is transitive, distributive, and also trivial over the unit of I. The first example of a pseudo commutative monoid with cancellation is given by the worldsheets, with $X : I^2 \longrightarrow Cat$ defined as in Example 5.2 without the collarings and decompositions.

The pseudo commutative monoids with cancellation

$$C(\mathcal{M}) : I^2 \longrightarrow Cat$$

$$C(\mathcal{M}, H) : I^2 \longrightarrow Cat$$

of [6] are also needed for a rigorous definition of conformal field theory with general modular functor. Let $\mathbb{C}_2$ denote the pseudo semiring of finite dimensional complex vector spaces and let $\mathbb{C}_2^{\text{Hilb}}$ denote the pseudo $\mathbb{C}_2$-algebra of complex separable Hilbert spaces with the operation $\hat{\otimes}$ of Hilbert tensor product. If $\mathcal{M}$ is a free finitely generated pseudo module over $\mathbb{C}_2$, then a map $H : \mathbb{C}_2 \longrightarrow \mathcal{M} \otimes_{\mathbb{C}_2} \mathbb{C}_2^{\text{Hilb}}$ of pseudo $\mathbb{C}_2$-modules is a Hilbert space if $\mathcal{M}$ is free of rank 1, otherwise such a map is a collection of objects indexed by a basis of $\mathcal{M}$.

Definition 6.1. If $\mathcal{M}$ is a free finitely generated pseudo module over $\mathbb{C}_2$ and I denotes the category of finite sets and bijections, then

$$C(\mathcal{M}) : I^2 \longrightarrow Cat$$

is the pseudo commutative monoid with cancellation defined by

$$C(\mathcal{M})_{a,b} = \mathcal{M}^{*\otimes a} \otimes \mathcal{M}^{\otimes b}$$

where $\mathcal{M}^* = \mathrm{Hom}_{\mathrm{pseudo}}(\mathcal{M}, \mathbb{C}_2)$. The operation $+$ is given by $\otimes$ and cancellation $\check{?}$ is given by evaluation $tr : \mathcal{M}^* \otimes \mathcal{M} \longrightarrow \mathbb{C}_2$.

Definition 6.2. If $\mathcal{M}$ is a free finitely generated pseudo module over $\mathbb{C}_2$, $H : \mathbb{C}_2 \longrightarrow \mathcal{M} \otimes_{\mathbb{C}_2} \mathbb{C}_2^{\mathrm{Hilb}}$ is a map of pseudo $\mathbb{C}_2$-modules, and I denotes the category of finite sets and bijections, then

$$C(\mathcal{M}, H) : I^2 \longrightarrow Cat$$

is the pseudo commutative monoid with cancellation such that an object of

$$C(\mathcal{M}, H)_{a,b}$$

is an object M of $C(\mathcal{M})_{a,b}$ equipped with a morphism

$$M \longrightarrow H^{*\hat{\otimes} a} \hat{\otimes} H^{\hat{\otimes} b}$$

in $C(\mathcal{M}, H)_{a,b}$ whose image consists of *trace class elements*. The operation $+$ is given by $\otimes$ and $\hat{\otimes}$, and $\check{?}$ is given by trace appropriately.

Definition 6.3. If $H_1, \ldots, H_n$ are complex separable Hilbert spaces, then an element x of the Hilbert tensor product $H_1 \hat{\otimes} \cdots \hat{\otimes} H_n$ is called *trace class* if there exist unit vectors $e_{ij} \in H_j$ and complex numbers μ_i such that

$$x = \sum_i \mu_i (e_{i1} \otimes \cdots \otimes e_{\mathrm{in}}) \quad \text{and}$$

$$\sum_i |\mu_i| < \infty.$$

Definition 6.4. Let X be the pseudo commutative monoid of worldsheets. A *modular functor with labels $\mathcal{M}$* is a morphism $\phi : X \longrightarrow C(\mathcal{M})$ of stacks of pseudo commutative monoids with cancellation. A *conformal field theory with modular functor on labels $\mathcal{M}$ with state space H* is a morphism $\Phi : X \longrightarrow C(\mathcal{M}, H)$ of stacks of pseudo commutative monoids with cancellation.

One can also take X to be other pseudo commutative monoids, for example, the pseudo commutative monoid of Jacobians with boundary as in [4].

6.2. The cobordism approach to conformal field theory

As we have seen in Theorems 5.3 and 5.4, a pseudo algebra over the cobordism 2-theory gives rise to a pseudo commutative monoid with cancellation. Still more is true. A conformal field theory in the cobordism approach gives rise to a conformal field theory in the commutative monoid with cancellation approach. We only need one concept to define conformal field theory in the cobordism approach.

Definition 6.5. A *strict algebra* over the 2-*theory of monoidal composition* consists of a commutative monoid $(I, +)$ and a function

$$X : I^2 \longrightarrow Sets$$

with three natural operations

$$X_{A,B} \times X_{C,D} \xrightarrow{+} X_{A+C,B+D}$$

$$\{*\} \xrightarrow{0} X_{0,0}$$

$$X_{B,C} \times X_{A,B} \xrightarrow{\circ} X_{A,C}$$

such that axioms (1)–(4) of Definition 5.1 hold.

Lemma 6.6. *Every pseudo commutative monoid with cancellation has an underlying pseudo algebra over the 2-theory of monoidal composition. Every pseudo algebra over the cobordism 2-theory has an underlying pseudo algebra over the 2-theory of monoidal composition.*

Proof. If X is a pseudo commutative monoid with cancellation, the composition is defined by

$$X_{B,C} \times X_{A,B} \xrightarrow{+} X_{B+A,C+B} \xrightarrow{\cong} X_{A+B,C+B} \xrightarrow{\check{?}} X_{A,C}.$$

If X is a pseudo algebra over the cobordism 2-theory, then one simply forgets the annuli structure. $\square$

Definition 6.7. Let X be the pseudo algebra over the cobordism 2-theory consisting of worldsheets. A *modular functor with labels $\mathcal{M}$ in the cobordism approach* is a morphism $\phi : X \longrightarrow C(\mathcal{M})$ of stacks of the underlying pseudo algebras over the 2-theory of monoidal composition. A *conformal field theory with modular functor on labels $\mathcal{M}$ with state space H in the cobordism approach* is a morphism $\Phi : X \longrightarrow C(\mathcal{M}, H)$ of stacks of the underlying pseudo algebras over the 2-theory of monoidal composition.

More generally, one could replace the pseudo algebra X with another pseudo algebra over the cobordism 2-theory.

Theorem 6.8. *Every modular functor and every conformal field theory in the cobordism approach gives rise to a modular functor and a conformal field theory in the commutative monoid with cancellation approach.*

Proof. Let X be a pseudo algebra over the cobordism 2-theory, and $\Phi : X \longrightarrow C(\mathcal{M}, H)$ a conformal field theory in the cobordism approach. As we have seen in Theorems 5.3 and 5.4, X gives rise to a pseudo commutative monoid, which we also denote X. Since Φ already preserves $+$ and 0 up to coherence iso, we only need to show that $\Phi(\check{x})$ is coherently isomorphic to $\Phi(\check{x})$. This follows from the transitivity of gluing and axioms (9) and (12) of Definition 5.1. $\square$

Remark 6.9. In the cobordism approach, one often defines a conformal field theory as an assignment which takes circles to Hilbert spaces and worldsheets to Hilbert–Schmidt operators. Such a conformal field theory gives rise to a conformal field theory in the cobordism approach defined above: a worldsheet with collaring is mapped to a composition of Hilbert–Schmidt operators, which is in turn a trace class element of the Hilbert tensor product.[5] Hence a conformal field theory in the cobordism approach defined using Hilbert–Schmidt operators gives rise to a conformal field theory in the commutative monoid with cancellation approach by Theorem 6.8.

7. Pseudo 2-algebras

Theorem 3.7 is a special case of the more general Theorem 7.9 about another new algebraic structure on a bicategory called *pseudo 2-algebra over a theory T with adjoined operations*. The adjoined operations are needed for the Frobenius elements. The proof of Theorem 7.9 is also the model for the proof of Theorem 5.7, which is the analogue of Theorem 3.7 for pseudo algebras over the cobordism 2-theory. In this section, we define this new algebraic structure on a bicategory and prove the relevant theorems. First we introduce pseudo 2-algebras over a theory T without the adjoined operations and prove the unadjoined Theorem 7.5 before moving on to the adjoined version in Theorem 7.9. Appendix A contains definitions of the various bicategorical notions appearing in this section.

Definition 7.1. Let T be a theory. A *pseudo 2-algebra over T with underlying pseudo T-algebra I* consists of

- a bicategory $\mathcal{C}$ with Obj $\mathcal{C}$ = Obj I,
- a pseudo T-algebra I, all of whose morphisms are invertible,
- a strict 2-functor $P : I \longrightarrow \mathcal{C}$ which is the identity on objects,
- structure maps $\Phi_n^{\mathcal{C}} : T(n) \longrightarrow \mathrm{End}_{\mathrm{pseudo}}(\mathcal{C})(n)$ where $\mathrm{End}_{\mathrm{pseudo}}(\mathcal{C})(n)$ denotes the set of pseudo functors $\mathcal{C}^n \longrightarrow \mathcal{C}$, and $P \circ \Phi^I(w) = \Phi^{\mathcal{C}}(w)(P, \dots, P)$ for all $w \in T(m)$,

[5] See [12] for trace class operators.

- the P-images of the natural coherence isos c, I, s of the pseudo T-algebra[6] I are pseudo natural coherence isos for each operation of theories:
  (1) For every $k \in \mathbb{N}$, $w \in T(k)$, and words $w_1, \dots, w_k$, we have a pseudo natural isomorphism $c_{w,w_1,\dots,w_k} : \Phi^{\mathcal{C}}(\gamma(w, w_1, \dots, w_k)) \Longrightarrow \gamma(\Phi^{\mathcal{C}}(w), \Phi^{\mathcal{C}}(w_1), \dots, \Phi^{\mathcal{C}}(w_k))$.
  (2) We have a pseudo natural isomorphism $I : \Phi^{\mathcal{C}}(1) \Longrightarrow 1_X$ where 1 is the identity word of the theory T and 1_X is the identity functor $X \longrightarrow X$.
  (3) For every word $w \in T(m)$ and function $f : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$, we have a pseudo natural isomorphism $s_{w,f} : \Phi^{\mathcal{C}}(w_f) \Longrightarrow \Phi^{\mathcal{C}}(w)_f$.
  (4) Consider the inscribed cubes whose front and back faces are the P-images of the coherence diagrams for c, I, s of the pseudo T-algebra I and the edges connecting the front face to the back face come from morphisms of $\mathcal{C}$. The pseudo naturality coherence iso 2-cells for $c_{w,w_1,\dots,w_k}$, I, and $s_{w,f}$ are coherent with each other in the sense that the 2-cells in this cube commute. These cubes are analogous to the cubes in (1)–(6) in Theorem 3.7.

Definition 7.2. Let $P : I \longrightarrow \mathcal{C}$ and $P' : I \longrightarrow \mathcal{C}'$ be pseudo 2-algebras over T, each with underlying pseudo T-algebra I. A *morphism* $P \longrightarrow P'$ *of pseudo 2-algebras over* T *with underlying pseudo* T*-algebra* I is a pseudo functor $G : \mathcal{C} \longrightarrow \mathcal{C}'$ such that

$$\begin{tikzcd} I \arrow[r, "P"] \arrow[dr, "P'"'] & \mathcal{C} \arrow[d, "G"] \\ & \mathcal{C}' \end{tikzcd}$$

strictly commutes. Furthermore, for each $w \in T(n)$ and for n-tuples $\overline{A}$ and $\overline{B}$ of objects of $\mathcal{C}$, the pseudo functor G is equipped with coherence natural iso 2-cells $\rho_w^{\overline{A},\overline{B}}$

$$\begin{tikzcd} \overline{\mathcal{C}(A,B)} \arrow[r, "\Phi^{\mathcal{C}}(w)"] \arrow[d, "\overline{G_{A,B}}"'] & \mathcal{C}(\Phi^{\mathcal{C}}(w)(\overline{A}), \Phi^{\mathcal{C}}(w)(\overline{B})) \arrow[d, "G_{\Phi^{\mathcal{C}}(w)(\overline{A}), \Phi^{\mathcal{C}}(w)(\overline{B})}"] \arrow[dl, Rightarrow, "\rho_w^{\overline{A},\overline{B}}"'] \\ \overline{\mathcal{C}'(A,B)} \arrow[r, "\Phi'^{\mathcal{C}}(w)"'] & \mathcal{C}'(\Phi'^{\mathcal{C}}(w)(\overline{A}), \Phi'^{\mathcal{C}}(w)(\overline{B})) \end{tikzcd}$$

which commute with the pseudo naturality coherence 2-cells of $c_{w,w_1,\dots,w_k}$, I, and $s_{w,f}$ of Definition 7.1.

Remark 7.3. From Definition 7.2 it follows that any morphism G is the identity on objects and is a strict 2-functor on the image of P. In particular G preserves identity morphisms.

Definition 7.4. If G and H are morphisms $P \longrightarrow P'$ as in Definition 7.2, then a *2-cell* $\sigma : G \Longrightarrow H$ consists of natural transformations $\sigma_{A,B} : G_{A,B} \Longrightarrow H_{A,B}$ for all $A, B \in \mathrm{Obj}\ \mathcal{C}$ compatible with $(\rho^G)_w^{\overline{A},\overline{B}}$ and $(\rho^H)_w^{\overline{A},\overline{B}}$. We also require further that

$$(i_{P'(k)} * \sigma_{A,B}^f) * i_{P'(j^{-1})} = \sigma_{C,D}^{(P(k)\circ f)\circ P(j^{-1})}$$

for all $f \in \mathcal{C}(A, B)$, $j \in I(A, C)$, and $k \in I(B, D)$.

Theorem 7.5. *Let* T *be a theory and* I *a pseudo* T*-algebra. Let* $\mathcal{C}_{T,I}$ *denote the 2-category of pseudo 2-algebras over* T *with underlying pseudo* T*-algebra* I *and strict identity morphisms. Then* $\mathcal{C}_{T,I}$ *is biequivalent to the 2-category*

[6] Unfortunately the letter I is overused here. It will be clear from the context whether I means the underlying pseudo algebra, or the coherence iso for the identity.

$\mathcal{C}_{\Theta,T,I}$ of pseudo algebras over the following 2-theory (Θ, T) with underlying groupoid I and strict identities. We describe the 2-theory (Θ, T) in terms of a strict (Θ, T)-algebra $X : I^2 \longrightarrow Set$. There are category-type operations

$$X_{B,C} \times X_{A,B} \xrightarrow{\circ} X_{A,C} \tag{5}$$

$$\{*\} \xrightarrow{1_B} X_{B,B} \tag{6}$$

called composition and unit and for every word $w \in T(n)$ there is an operation

$$X_{A_1,B_1} \times \cdots \times X_{A_n,B_n} \longrightarrow X_{w(A_1,\ldots,A_n),w(B_1,\ldots,B_n)}. \tag{7}$$

These operations satisfy the relations of a theory (associativity, unitality, equivariance, functoriality) and are compatible with $\circ$ and 1_B.

Proof. We begin by describing a strict 2-functor $K : \mathcal{C}_{\Theta,T,I} \longrightarrow \mathcal{C}_{T,I}$. Let $X : I^2 \longrightarrow Cat$ be a pseudo algebra over the 2-theory (Θ, T) with underlying pseudo T-algebra I. Let $\mathcal{C}$ be the bicategory with objects Obj I and morphism categories $\mathcal{C}(A, B) := X_{A,B}$. Define a strict 2-functor $P : I \longrightarrow \mathcal{C}$ which is the identity on objects and does

$$f \mapsto X_{1^v_A, f}(1_A) \in X_{A,B}$$

for morphisms $f : A \longrightarrow B$ of I, where 1^v_A denotes the identity on A in I. For $A \xrightarrow{f} B \xrightarrow{g} C$ we have

$$\begin{aligned} P(g \circ f) &= X_{1^v_A, g\circ f}(1_A) \\ &= X_{1^v_A, g}(X_{1^v_A, f}(1_A)) \\ &= X_{1^v_A, g}(P(f)) \\ &= X_{1^v_A, g}(1_B \circ P(f)) \\ &= X_{1^v_B, g}(1_B) \circ P(f) \\ &= P(g) \circ P(f) \end{aligned}$$

by the commutative diagram below.

$$\begin{tikzcd} X_{B,B} \times X_{A,B} \arrow[r, "\circ"] \arrow[d, "X_{1^v_B,g}"'] \arrow[d, shift left=8, "X_{1^v_A,1^v_B}"] & X_{A,B} \arrow[d, "X_{1^v_A,g}"] \\ X_{B,C} \times X_{A,B} \arrow[r, "\circ"'] & X_{A,C} \end{tikzcd}$$

The 2-functor P preserves identity morphisms because

$$\begin{aligned} P(1^v_A) &= X_{1^v_A, 1^v_A}(1_A) \\ &= 1_A. \end{aligned}$$

This defines the 2-functor P of $K(X)$. Next we need structure maps $\Phi^{\mathcal{C}}_n : T(n) \longrightarrow \mathrm{End}_{\mathrm{pseudo}}(\mathcal{C})(n)$.

Let $\Phi^I_n : T(n) \longrightarrow \mathrm{End}(I)(n)$ denote the structure maps for the pseudo T-algebra I. For $w \in T(n)$ define $\Phi^{\mathcal{C}}_n(w) : \mathcal{C}^n \longrightarrow \mathcal{C}$ on objects as $\Phi^I_n(w)$. On the hom categories $\mathcal{C}(A, B)$ define $\Phi^{\mathcal{C}}_n(w)$ by (7). The pseudo 2-algebra coherence isos $c_{w,w_1,\ldots,w_k}$, I, and $s_{w,f}$ for $\mathcal{C}$ are obtained by applying P to the coherence isos of the pseudo T-algebra I. The coherence iso 2-cells for the pseudo naturality of $c_{w,w_1,\ldots,w_k}$, I, and $s_{w,f}$ for $\mathcal{C}$ arise from the coherence iso modifications of X. The pseudo 2-algebra diagrams commute because the pseudo (Θ, T)-algebra diagrams commute. Hence $K(X) \in \mathcal{C}_{T,I}$.

Let $G : X \longrightarrow X'$ be a morphism in $\mathcal{C}_{\Theta,T,I}$, *i.e.* G is a strict 2-natural transformation from X to X' which preserves the structure maps of X and X' up to coherence iso modifications which satisfy coherence diagrams and G preserves the identities strictly. Let $P' = K(X')$. Define the pseudo functor $K(G) : \mathcal{C} \longrightarrow \mathcal{C}'$ to be the identity on objects, and as $G_{A,B} : X_{A,B} \longrightarrow X'_{A,B}$ on $\mathcal{C}(A, B) = X_{A,B}$. Since G is natural and preserves identity morphisms, we have for $f : A \longrightarrow B$

$$\begin{aligned} K(G)P(f) &= G_{A,B}(P(f)) \\ &= G_{A,B}(X_{1^v_A,f}(1_A)) \\ &= X'_{1^v_a,f}(G_{A,A}(1_A)) \\ &= X'_{1^v_a,f}(1_A) \\ &= P'(f). \end{aligned}$$

The coherences isos of $K(G)$ commute with the coherence isos of $K(X)$ and $K(X')$ because the coherence isos of G commute with the coherence isos of X and X'. Hence $K(G)$ is a morphism in $\mathcal{C}_{T,I}$.

For a 2-cell $\sigma : G \Longrightarrow H$ in $\mathcal{C}_{\Theta,T,I}$, *i.e.* for a modification $\sigma : G \rightsquigarrow H$ compatible with the coherence iso modifications of G and H, the natural transformation $K(\sigma)_{A,B}$ is simply $\sigma_{A,B}$. One can easily check that K so defined strictly preserves compositions and preserves identity morphisms. Hence $K : \mathcal{C}_{\Theta,T,I} \longrightarrow \mathcal{C}_{T,I}$ is a strict 2-functor.

Next we describe a strict 2-functor $L : \mathcal{C}_{T,I} \longrightarrow \mathcal{C}_{\Theta,T,I}$. Let $P : I \longrightarrow \mathcal{C}$ be a pseudo 2-algebra over T with underlying pseudo T-algebra I. We define a strict 2-functor $L(P) := X : I^2 \longrightarrow Cat$ as follows. For an object $(A, B) \in I^2$, the objects of $X_{A,B}$ are tuples $f = (f_m, \dots, f_1)$ for $m \geq 1$

$$A = A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} A_{m-1} \xrightarrow{f_m} A_m = B$$

such that f_1 and f_m are in the image of P and all f_j are morphisms in $\mathcal{C}$. In particular, we also include $P(h)$ in $X_{A,B}$ for all h in $I(A, B)$. Note that the image of P is actually a strict 2-category because P is a strict 2-functor. A morphism $\phi : (f_m, \dots, f_1) \longrightarrow (g_n, \dots, g_1)$ in $X_{A,B}$ is a 2-cell

$$f_m \circ (f_{m-1} \circ (\cdots \circ (f_2 \circ f_1))) \Longrightarrow g_n \circ (g_{n-1} \circ (\cdots \circ (g_2 \circ g_1)))$$

in $\mathcal{C}$. For a morphism $(j, k) : (A, B) \longrightarrow (C, D)$ in I^2, the functor

$$X_{j,k} : X_{A,B} \longrightarrow X_{C,D}$$

is defined by $(f_m, \dots, f_1) \longmapsto (P(k) \circ f_m, \dots, f_1 \circ P(j^{-1}))$. These assignments make $X : I^2 \longrightarrow Cat$ into a strict 2-functor because the image of P is strict.

The composition in (5) is defined by

$$\begin{aligned} &X_{B,C} \times X_{A,B} \xrightarrow{\circ} X_{A,C} \\ &((g_n, \dots, g_1), (f_m, \dots, f_1)) \longmapsto (g_n, \dots, g_2, g_1 \circ f_m, f_{m-1}, \dots, f_2, f_1). \end{aligned}$$

This composition is strictly unital with identity $P(1_B) \in X_{B,B}$ and is natural for morphisms of I.

As part of the pseudo 2-algebra structure on $P : I \longrightarrow \mathcal{C}$ we have maps $\Phi^{\mathcal{C}}_n : T(n) \longrightarrow \mathrm{End}_{\mathrm{pseudo}}(\mathcal{C})(n)$, which give rise to the pseudo (Θ, T)-operations in (7) as follows. For simplicity we explicitly define them only for $n = 2$. For $w \in T(2)$, we define

$$\begin{aligned} &X_{A,B} \times X_{C,D} \longrightarrow X_{\Phi^I_2(w)(A,C),\, \Phi^I_2(w)(B,D)} \\ &((f_m, \dots, f_1), (g_m, \dots, g_1)) \longmapsto (\Phi^{\mathcal{C}}_2(w)(f_m, g_m), \dots, \Phi^{\mathcal{C}}_2(w)(f_1, g_1)) \end{aligned}$$

on objects of $X_{A,B} \times X_{C,D}$ and similarly on morphisms. The pseudo (Θ, T)-algebra coherence isos come from the bicategory coherence isos of $\mathcal{C}$ and the pseudo naturality coherences iso 2-cells of $c_{w,w_1,\dots,w_k}$, I, and $s_{w,f}$. This concludes the definition of $L(P)$.

If $G : P \longrightarrow P'$ is a morphism in $\mathcal{C}_{T,I}$, then $L(G)_{A,B} : X_{A,B} \longrightarrow X'_{A,B}$ is defined componentwise. Similarly, if $\sigma : G \Longrightarrow H$ is a 2-cell in the $\mathcal{C}_{T,I}$, then for $f = (f_m, \dots, f_1) \in X_{A,B}$, the morphism $(L(\sigma)_{A,B})_f$ of $X'_{A,B}$ is

$$\begin{array}{c}
G(f_m) \circ (G(f_{m-1}) \circ (\cdots \circ (G(f_2) \circ G(f_1)))) \\
\Big\Downarrow \cong \\
G(f_m \circ (f_{m-1} \circ (\cdots \circ (f_2 \circ f_1)))) \\
\Big\Downarrow \sigma_{f_m \circ (f_{m-1} \circ (\cdots \circ (f_2 \circ f_1)))} \\
H(f_m \circ (f_{m-1} \circ (\cdots \circ (f_2 \circ f_1)))) \\
\Big\Downarrow \cong \\
H(f_m) \circ (H(f_{m-1}) \circ (\cdots \circ (H(f_2) \circ H(f_1))))
\end{array}$$

These assignments for L clearly preserve composition and identity morphisms because L is defined componentwise. Hence we have defined a strict 2-functor $L : \mathcal{C}_{T,I} \longrightarrow \mathcal{C}_{\Theta,T,I}$.

The objects $KL(P)$ and P are equivalent under the pseudo functor which is the identity on objects and

$$(f_m, \dots, f_1) \longmapsto f_m \circ (f_{m-1} \circ (\cdots \circ (f_2 \circ f_1)))$$

on morphisms. Hence the 2-functor KL is pseudo equivalent to $1_{\mathcal{C}_{T,I}}$. Similarly, LK is pseudo equivalent to $1_{\mathcal{C}_{\Theta,T,I}}$, and finally K and L are biequivalences. $\square$

Unfortunately, pseudo 2-algebras over a theory T are not enough to capture Frobenius elements. Therefore we introduce adjoined operations now. Theorem 7.9, which has Theorem 3.7 as a corollary, is an adjoined version of Theorem 7.5.

Definition 7.6. Let $P : I \longrightarrow \mathcal{C}$ be a pseudo 2-algebra over T. Then an *adjoined operation with source* $w_1, \dots, w_q \in T^2(m)$ *and target* $w \in T^2(m)$ is a functor

$$\mathcal{C}(\Phi^{\mathcal{C}}(w_1^1)(\overline{A}), \Phi^{\mathcal{C}}(w_1^2)(\overline{A})) \times \cdots \times \mathcal{C}(\Phi^{\mathcal{C}}(w_q^1)(\overline{A}), \Phi^{\mathcal{C}}(w_q^2)(\overline{A})) \longrightarrow \mathcal{C}(\Phi^{\mathcal{C}}(w^1)(\overline{A}), \Phi^{\mathcal{C}}(w^2)(\overline{A}))$$

for each m-tuple $\overline{A}$ of objects of $\mathcal{C}$, which is strictly natural for m-tuples $\overline{k}$ of maps in I. For example, for an adjoined operation with source $w_1 \in T^2(m)$ the diagram

$$\begin{tikzcd}
\mathcal{C}(\Phi^{\mathcal{C}}(w_1^1)(\overline{A}), \Phi^{\mathcal{C}}(w_1^2)(\overline{A})) \arrow[r] \arrow[d, "\Phi^{\mathcal{C}}(w_1^2)(P(\overline{k}))\circ - \circ \Phi^{\mathcal{C}}(w_1^1)(P(\overline{k})^{-1})"'] & \mathcal{C}(\Phi^{\mathcal{C}}(w^1)(\overline{A}), \Phi^{\mathcal{C}}(w^2)(\overline{A})) \arrow[d, "\Phi^{\mathcal{C}}(w^2)(P(\overline{k}))\circ - \circ \Phi^{\mathcal{C}}(w^1)(P(\overline{k})^{-1})"] \\
\mathcal{C}(\Phi^{\mathcal{C}}(w_1^1)(\overline{A'}), \Phi^{\mathcal{C}}(w_1^2)(\overline{A'})) \arrow[r] & \mathcal{C}(\Phi^{\mathcal{C}}(w^1)(\overline{A'}), \Phi^{\mathcal{C}}(w^2)(\overline{A'}))
\end{tikzcd}$$

must commute.

Definition 7.7. A pseudo 2-algebra over a theory T *with adjoined operations* is a pseudo 2-algebra $\mathcal{C}$ equipped with specified adjoined operations and specified natural coherence iso 2-cells between compositions of adjoined operations. These coherence iso 2-cells are coherent with the coherence iso 2-cells of $c_{w,w_1,\dots,w_k}$, I, and $s_{w,f}$.

Example 7.8. A pseudo Frobenius symmetric monoidal category gives rise to a pseudo T-algebra with adjoined operations. The adjoined operations are

$$* \xrightarrow{m_A} \mathcal{C}(0, A + A)$$

$$* \xrightarrow{n_A} \mathcal{C}(A + A, 0)$$

and T is the theory of commutative monoids. The natural coherence iso 2-cells and their diagrams are listed at the end of Theorem 3.7.

Theorem 7.9. *Let (Θ, T) be a 2-theory which has category operations*

$$X_{B,C} \times X_{A,B} \xrightarrow{\circ} X_{A,C}$$

$$\{*\} \xrightarrow{1_B} X_{B,B}$$

and for every word $w \in T(n)$ there is an operation

$$X_{A_1,B_1} \times \cdots \times X_{A_n,B_n} \longrightarrow X_{w(A_1,\ldots,A_n),w(B_1,\ldots,B_n)}$$

and these operations satisfy the diagrams of a theory (composition, unit, substitution) and are compatible with $\circ$ and 1_B. Then the 2-category $\mathcal{C}_{\Theta,T,I}$ of pseudo (Θ, T)-algebras with underlying groupoid I and strict identity morphisms is biequivalent to the 2-category $\mathcal{C}^{\mathrm{adj}}_{T,I}$ of pseudo 2-algebras $P : I \longrightarrow \mathcal{C}$ over T with an adjoined operation

$$\mathcal{C}(\Phi^{\mathcal{C}}(w_1^1)(\overline{A}),\ \Phi^{\mathcal{C}}(w_1^2)(\overline{A})) \times \cdots \times \mathcal{C}(\Phi^{\mathcal{C}}(w_q^1)(\overline{A}),\ \Phi^{\mathcal{C}}(w_q^2)(\overline{A})) \longrightarrow \mathcal{C}(\Phi^{\mathcal{C}}(w^1)(\overline{A}),\ \Phi^{\mathcal{C}}(w^2)(\overline{A}))$$

for each element of $\Theta(m)(w; w_1, \ldots, w_q)$ and a natural coherence iso 2-cell for each operation of 2-theories satisfying coherence diagrams for each relation of theories.

Proof. This proof builds on the proof of Theorem 7.5. The 2-functors $K : \mathcal{C}_{\Theta,T,I} \longrightarrow \mathcal{C}^{\mathrm{adj}}_{T,I}$ and $L : \mathcal{C}^{\mathrm{adj}}_{T,I} \longrightarrow \mathcal{C}_{\Theta,T,I}$ are defined as in Theorem 7.5. For an object X of $\mathcal{C}_{\Theta,T,I}$, the adjoined operations on $K(X)$ come directly from the structure maps of X because $K(X)(A, B)$ is $X_{A,B}$. The coherence diagrams for $K(X)$ commute because they are precisely the coherence diagrams for X. On the other hand, for an object P of $\mathcal{C}^{\mathrm{adj}}_{T,I}$, the structure maps of $L(P)$ are defined by first composing the tuples $(f_m, \ldots, f_1)$, then applying the adjoined operations to composed tuples $f_m \circ (f_{m-1} \circ (\cdots \circ (f_2 \circ f_1)))$, and finally placing the result r into $(1_{\mathrm{range}\ r}, r, 1_{\mathrm{domain}\ r})$. We obtain a biequivalence by the argument of Theorem 7.5. □

Acknowledgements

The author was supported by the Mathematical Sciences Postdoctoral Research Fellowship of the National Science Foundation.

The author extends his gratitude to Igor Kriz for helpful discussions on this topic, and also to Peter May, Craig Westerland, and the referee for comments on an earlier version of this paper.

Appendix A. Bicategories, theories, 2-theories, and pseudo algebras

The basics of bicategories can be found in the original paper [1], in the concise account [9], or in the last chapter of [10]. A thorough description of theories, 2-theories, and pseudo algebras can be found in [3]. Theories were introduced in [8], while 2-theories and pseudo algebras over theories and 2-theories were introduced in [5]. A review of the main definitions and examples of bicategories, theories, and 2-theories is below for the reader's convenience.

Definition A.1. A *bicategory* $\mathcal{C}$ is comprised of a collection Obj $\mathcal{C}$ of *objects* $A, B, C \ldots$, categories $\mathrm{Mor}_{\mathcal{C}}(A, B)$, functors

$$\mathrm{Mor}_{\mathcal{C}}(B, C) \times \mathrm{Mor}_{\mathcal{C}}(A, B) \xrightarrow{\circ} \mathrm{Mor}_{\mathcal{C}}(A, C)$$

$$\{*\} \xrightarrow{1_A} \mathcal{C}(A, A),$$

and natural isomorphisms called *coherence isomorphisms*

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(C, D) \times \mathrm{Mor}_{\mathcal{C}}(B, C) \times \mathrm{Mor}_{\mathcal{C}}(A, B) \arrow[r, "\circ \times 1_{\mathrm{Mor}_{\mathcal{C}}(A,B)}"] \arrow[d, "1_{\mathrm{Mor}_{\mathcal{C}}(C,D)} \times \circ"'] & \mathrm{Mor}_{\mathcal{C}}(B, D) \times \mathrm{Mor}_{\mathcal{C}}(A, B) \arrow[d, "\circ"] \\
\mathrm{Mor}_{\mathcal{C}}(C, D) \times \mathrm{Mor}_{\mathcal{C}}(A, C) \arrow[r, "\circ"'] \arrow[ur, Rightarrow, "\alpha^{\mathcal{C}}_{A,B,C}"] & \mathrm{Mor}_{\mathcal{C}}(A, D)
\end{tikzcd}$$

$$\begin{tikzcd}
\{*\} \times \mathrm{Mor}_{\mathcal{C}}(A,B) \arrow[r, "\cong"] \arrow[d, "1_B \times 1_{\mathrm{Mor}_{\mathcal{C}}(A,B)}"'] & \mathrm{Mor}_{\mathcal{C}}(A,B) \arrow[d, equal] \\
\mathrm{Mor}_{\mathcal{C}}(B,B) \times \mathrm{Mor}_{\mathcal{C}}(A,B) \arrow[r, "\circ"'] \arrow[ur, Rightarrow, "\lambda_{A,B}"] & \mathrm{Mor}_{\mathcal{C}}(A,B)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(A,B) \times \{*\} \arrow[r, "\cong"] \arrow[d, "1_{\mathrm{Mor}_{\mathcal{C}}(A,B)} \times 1_A"'] & \mathrm{Mor}_{\mathcal{C}}(A,B) \arrow[d, equal] \\
\mathrm{Mor}_{\mathcal{C}}(A,B) \times \mathrm{Mor}_{\mathcal{C}}(A,A) \arrow[r, "\circ"'] \arrow[ur, Rightarrow, "\rho_{A,B}"] & \mathrm{Mor}_{\mathcal{C}}(A,B),
\end{tikzcd}$$

such that the following *coherence diagrams* are satisfied.

(1) For composable e, f, g, and h the associativity pentagon

$$\begin{tikzcd}
h(g(fe)) \arrow[r, Rightarrow, "i_h * \alpha_{g,f,e}"] \arrow[d, Rightarrow, "\alpha_{h,g,(fe)}"'] & h((gf)e) \arrow[dd, Rightarrow, "\alpha_{h,(gf),e}"] \\
(hg)(fe) \arrow[d, Rightarrow, "\alpha_{(hg),f,e}"'] & \\
((hg)f)e & (h(gf))e \arrow[l, Rightarrow, "\alpha_{h,g,f} * i_e"]
\end{tikzcd}$$

commutes.

(2) For $A \xrightarrow{f} B \xrightarrow{g} C$, the left and right identity coherence 2-cells commute.

$$\begin{tikzcd}
g(1_B f) \arrow[r, Rightarrow, "\alpha^{\mathcal{C}}_{g,1_B,f}"] \arrow[d, Rightarrow, "i_g * \lambda^{\mathcal{C}}_f"'] & (g1_B)f \arrow[d, Rightarrow, "\rho^{\mathcal{C}}_g * i_f"] \\
gf \arrow[r, equal] & gf
\end{tikzcd}$$

The objects and morphisms of each category $\mathrm{Mor}_{\mathcal{C}}(A, B)$ are called *morphisms of* $\mathcal{C}$ and 2-*cells of* $\mathcal{C}$ respectively. Another common term for morphisms of $\mathcal{C}$ is 1-*cell*. The morphism category $\mathrm{Mor}_{\mathcal{C}}(A, B)$ is often written $\mathcal{C}(A, B)$. Morphisms of $\mathcal{C}$ are drawn with a single arrow, while 2-cells of $\mathcal{C}$ are drawn with a double arrow. The identity 2-cell on a morphism f is denoted i_f. The *horizontal composition* of 2-cells (given by $\circ$) is denoted by $\beta * \alpha$. The *vertical composition* of 2-cells (given by the composition in $\mathrm{Mor}_{\mathcal{C}}(A, B)$) is denoted $\beta \odot \alpha$.

Example A.2. A familiar algebraic example of a bicategory is the following. Objects are rings, morphisms from R to S are $S - R$ bimodules, and 2-cells are bimodule homomorphisms. Composition is given by tensor product of bimodules.

Definition A.3. A 2-*category* is a bicategory in which the coherence isomorphisms are identities.

Example A.4. The 2-category *Cat* has objects small categories, morphisms functors, and 2-cells natural transformations. This example is why natural transformations are often written as double arrows. Another example of 2-category is *Top*, which consists of topological spaces, continuous maps, and homotopy classes of homotopies. Any category can be regarded as a 2-category in which all the 2-cells are trivial.

Definition A.5. A *homomorphism of bicategories* $G : \mathcal{C} \longrightarrow \mathcal{C}'$ consists of a function

$$G : \mathrm{Obj}\, \mathcal{C} \longrightarrow \mathrm{Obj}\, \mathcal{C}',$$

functors

$$G_{A,B} : \mathrm{Mor}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Mor}_{\mathcal{C}'}(GA, GB),$$

and natural isomorphisms (*coherence isos*)

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(B, C) \times \mathrm{Mor}_{\mathcal{C}}(A, B) \arrow[r, "\circ"] \arrow[d, "G_{B,C} \times G_{A,B}"'] & \mathrm{Mor}_{\mathcal{C}}(A, C) \arrow[d, "G_{A,C}"] \\
\mathrm{Mor}_{\mathcal{C}'}(GB, GC) \times \mathrm{Mor}_{\mathcal{C}'}(GA, GB) \arrow[r, "\circ"'] \arrow[ur, Rightarrow, "\gamma_{A,B,C}"] & \mathrm{Mor}_{\mathcal{C}'}(GA, GC)
\end{tikzcd}$$

$$\begin{tikzcd}
\{*\} \arrow[r, "1_A"] \arrow[d, equal] & \mathrm{Mor}_{\mathcal{C}}(A, A) \arrow[d, "G_{A,A}"] \\
\{*\} \arrow[r, "1_{GA}"'] \arrow[ur, Rightarrow, "\delta_A"] & \mathrm{Mor}_{\mathcal{C}'}(GA, GA),
\end{tikzcd}$$

such that the *coherence diagrams*

$$\begin{tikzcd}
Gh \circ (Gf \circ Gf) \arrow[r, Rightarrow, "i_{Gh} * \gamma_{g,f}"] \arrow[d, Rightarrow, "\alpha^{\mathcal{C}'}_{Gh,Gg,Gf}"'] & Gh \circ G(g \circ f) \arrow[r, Rightarrow, "\gamma_{h,g \circ f}"] & G(h \circ (g \circ f)) \arrow[d, Rightarrow, "G(\alpha^{\mathcal{C}}_{h,g,f})"] \\
(Gh \circ Gg) \circ Gf \arrow[r, Rightarrow, "\gamma_{h,g} * i_{Gf}"'] & G(h \circ g) \circ Gf \arrow[r, Rightarrow, "\gamma_{h \circ g, f}"'] & G((h \circ g) \circ f)
\end{tikzcd}$$

$$\begin{tikzcd}
1_{GB} \circ Gf \arrow[r, Rightarrow, "\delta_B * i_{Gf}"] \arrow[d, Rightarrow, "\lambda_{GB}"'] & G(1_B) \circ Gf \arrow[d, Rightarrow, "\gamma_{1_B, f}"] \\
Gf & G(1_B \circ f) \arrow[l, Rightarrow, "G(\lambda_B)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Gf \circ 1_{GA} \arrow[r, Rightarrow, "i_{Gf} * \delta_A"] \arrow[d, Rightarrow, "\rho_{GA}"'] & Gf \circ G(1_A) \arrow[d, Rightarrow, "\gamma_{f, 1_A}"] \\
Gf & G(f \circ 1_A) \arrow[l, Rightarrow, "G(\rho_A)"]
\end{tikzcd}$$

commute. When $\mathcal{C}$ and $\mathcal{C}'$ are 2-categories, a homomorphism G is often called a *pseudo functor*. In this paper we use the terms synonymously. When $\mathcal{C}$ and $\mathcal{C}'$ are 2-categories and the coherence isomorphisms for G are trivial, then G is called a *(strict)* 2-*functor*.

Definition A.6. If $G, H : \mathcal{C} \longrightarrow \mathcal{C}'$ are homomorphisms of bicategories, then a *strong transformation* $\sigma : G \Longrightarrow H$ assigns to each $A \in \operatorname{Obj} \mathcal{C}$ a morphism $\sigma_A : GA \longrightarrow HA$ in $\mathcal{C}'$ and to each $A, B \in \operatorname{Obj} \mathcal{C}$ a natural isomorphism

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(A,B) \arrow[r, "G"] \arrow[d, "H"'] & \mathrm{Mor}_{\mathcal{C}'}(FA,FB) \arrow[d, "\sigma_B \circ"] \\
\mathrm{Mor}_{\mathcal{C}'}(GA,GB) \arrow[r, "\circ \sigma_A"'] \arrow[ur, Rightarrow, "\tau_{A,B}"] & \mathrm{Mor}_{\mathcal{C}'}(FA,GB)
\end{tikzcd}$$

such that the coherence diagrams

$$\begin{tikzcd}
(Hg \circ Hf) \circ \sigma_A \arrow[r, Rightarrow, "i_{Hg} * \tau_f"] \arrow[d, Rightarrow, "\gamma^H_{f,g} * i_{\sigma_A}"'] & Hg \circ \sigma_B \circ Gf \arrow[r, Rightarrow, "\tau_g * i_{Gf}"] & \sigma_C \circ (Gg \circ Gf) \arrow[d, Rightarrow, "i_{\sigma_C} * \gamma^G_{f,g}"] \\
H(g \circ f) \circ \sigma_A \arrow[rr, Rightarrow, "\tau_{g \circ f}"'] & & \sigma_C \circ G(g \circ f)
\end{tikzcd}$$

$$\begin{tikzcd}
\sigma_A \arrow[r, Rightarrow, "\lambda^H_A"] \arrow[d, Rightarrow, "\rho^G_A"'] & 1_{HA} \circ \sigma_A \arrow[r, Rightarrow, "\delta^H_A * i_{\sigma_A}"] & H(1_A) \circ \sigma_A \arrow[d, Rightarrow, "\tau_{1_A}"] \\
\sigma_A \circ 1_{GA} \arrow[rr, Rightarrow, "i_{\sigma_A} * \delta^G_A"'] & & \sigma_A \circ G(1_A)
\end{tikzcd}$$

commute, where we have suppressed three occurrences of $\alpha^{\mathcal{C}'}$ in the top row of the first diagram for legibility. The *vertical composition* of strong transformations is denoted $\sigma_2 \odot \sigma_1$. If $\mathcal{C}$ and $\mathcal{C}'$ are 2-categories, then a strong transformation is often called a *pseudo natural transformation*. If $\mathcal{C}$ and $\mathcal{C}'$ are 2-categories, G and H are 2-functors, and τ is the identity, then σ is called a 2-*natural transformation*.

Definition A.7. If $\sigma, \sigma' : G \Longrightarrow H$ are strong transformations, then a *modification* $\Xi : \sigma \rightsquigarrow \sigma'$ assigns to each object A of $\mathcal{C}$ a 2-cell $\Xi_A : \sigma_A \Longrightarrow \sigma'_A$ in $\mathcal{C}'$ such that

$$\tau^{\sigma'}_g \odot (H\beta * \Xi_A) = (\Xi_B * G\beta) \odot \tau^{\sigma}_f$$

for each 2-cell $\beta : f \Longrightarrow g$ and morphisms $f, g : A \longrightarrow B$ in $\mathcal{C}$.

Definition A.8. A homomorphism $G : \mathcal{C} \longrightarrow \mathcal{C}'$ of bicategories is a *biequivalence* if there exists a homomorphism $H : \mathcal{C}' \longrightarrow \mathcal{C}$, strong transformations

$$\begin{tikzcd}
HG \arrow[r, Rightarrow, shift left, "\varepsilon_1"] & 1_{\mathcal{C}} \arrow[l, Rightarrow, shift left, "\eta_1"]
\end{tikzcd} \qquad \begin{tikzcd}
GH \arrow[r, Rightarrow, shift left, "\varepsilon_2"] & 1_{\mathcal{C}'} \arrow[l, Rightarrow, shift left, "\eta_2"]
\end{tikzcd},$$

and iso modifications

$$\varepsilon_1 \odot \eta_1 \rightsquigarrow i_{1_{\mathcal{C}}} \qquad \eta_1 \odot \varepsilon_1 \rightsquigarrow i_{HG}$$

$$\varepsilon_2 \odot \eta_2 \rightsquigarrow i_{1_{\mathcal{C}'}} \qquad \eta_2 \odot \varepsilon_2 \rightsquigarrow i_{GH}.$$

Lemma A.9. *A homomorphism $G : \mathcal{C} \longrightarrow \mathcal{C}'$ of bicategories is a biequivalence if and only if*

- *For each object A' of $\mathcal{C}'$ there exists an object A of $\mathcal{C}$ such that GA is equivalent to A' in the sense that there are morphisms*

$$\begin{tikzcd} GA \arrow[r, bend left] & A' \arrow[l, bend left] \end{tikzcd}$$

in $\mathcal{C}'$ whose compositions are isomorphic to the respective identities, and

- *For all objects A, B in $\mathcal{C}$, the functor*

$$G_{A,B} : \mathrm{Mor}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Mor}_{\mathcal{C}'}(GA, GB)$$

is an equivalence of categories.

After this summary of bicategories, we now turn to theories and 2-theories.

Definition A.10. A *theory* is a category T with Obj $T = \{0, 1, 2, \ldots\}$ such that n is the product of 1 with itself n times in the category T and each n is equipped with projection maps that make it into a product. We use the notation $T(n) := \mathrm{Mor}_T(n, 1)$.

Example A.11. Let X be a set. Then the *endomorphism theory* $\mathrm{End}(X)$ has objects $0, 1, 2, \ldots$ and hom sets

$$\mathrm{Mor}_{\mathrm{End}(X)}(m, n) = \mathrm{Map}(X^m, X^n).$$

A theory can equivalently be described in terms of the sets $T(n)$ for $n \geq 0$ and *operations*:

- for all $k, n_1, \ldots, n_k \in \{0, 1, \ldots\}$ a map

$$\gamma : T(k) \times T(n_1) \times \ldots \times T(n_k) \longrightarrow T(n_1 + \cdots + n_k)$$

called *composition*,

- a *unit* $1 \in T(1)$,
- for every function $f : \{1, \ldots, k\} \longrightarrow \{1, \ldots, \ell\}$ a map

$$T(k) \xrightarrow{()f} T(\ell)$$

called *substitution*,

which satisfy the *relations*:

(1) composition is associative,
(2) composition is unital with unit 1,
(3) composition is equivariant with respect to substitution in two ways,
(4) substitution is functorial.

Definition A.12. Let X be a set and T a theory. Then X is a strict T*-algebra* if it is equipped with a morphism of theories $\Phi : T \longrightarrow \mathrm{End}(X)$. We also say X is a *strict algebra over the theory* T. The adjective strict is often left off.

In other words X is a T-algebra if it is equipped with a function $\Phi_n(w) : X^n \longrightarrow X$ for every *word* $w \in T(n)$ and this assignment Φ is compatible with composition, unit, and substitution. A *pseudo T-algebra* is like a T-algebra except that this assignment Φ is compatible with composition, unit, and substitution only up to a coherence isomorphism and these coherence isomorphisms satisfy coherence diagrams which come from the relations of theories.

Definition A.13. Let T be a theory. A category X is a *pseudo T-algebra or a pseudo algebra over T* if it is equipped with *structure maps* $\Phi_n : T(n) \longrightarrow \mathrm{Functors}(X^n, X)$ for every $n \geq 0$ as well as coherence isos for each operation of theories which satisfy coherence diagrams for each relation of theories. These diagrams are listed in [3].

Example A.14. The category of finite dimensional complex vector spaces with $\oplus$ and $\otimes$ is a pseudo algebra over the theory of commutative semirings.

A 2-theory is similar to a theory except that the words are indexed by words of an underlying theory.

Definition A.15. A 2-*theory Θ fibered over the theory T*, written (Θ, T) for short, is a natural number k, a theory T, and a contravariant functor $\Theta : T \longrightarrow Cat$ from the category T to the 2-category Cat of small categories such that:

- Obj $\Theta(m) = \coprod_{n\geq 0} \mathrm{Mor}_{T^k}(m,n)$ for all $m \in \mathbb{N}$, where T^k is the theory with the same objects as T, but with $\mathrm{Mor}_{T^k}(m,n) = \mathrm{Mor}_T(m,n)^k$,
- If $w_1, \ldots, w_n \in \mathrm{Mor}_{T^k}(m,1)$, then the word in $\mathrm{Mor}_{T^k}(m,n)$ with which the n-tuple $w_1, \ldots, w_n$ is identified is the product in $\Theta(m)$ of $w_1, \ldots, w_n$,
- For $w \in \mathrm{Mor}_T(m,n)$ the functor $\Theta(w) : \Theta(n) \longrightarrow \Theta(m)$ is $\Theta(w)(v) = v \circ w^{\times k}$ on objects $v \in \mathrm{Mor}_{T^k}(n,j)$.

For objects $w_1, \ldots, w_n, w \in \mathrm{Mor}_{T^k}(m,1) \subseteq$ Obj $\Theta(m)$ we set

$$\Theta(w; w_1, \ldots, w_n) := \mathrm{Mor}_{\Theta(m)}\left(\prod_{i=1}^{n} w_i, w\right).$$

Example A.16. Let I be a category and k a positive integer. Suppose $X : I^k \longrightarrow Cat$ is a strict 2-functor from the category I^k to the 2-category Cat of small categories. We will now describe the 2-*theory End(X) fibered over the theory End(I)*, which is a contravariant functor $\mathrm{End}(I) \longrightarrow Cat$ satisfying the axioms of a 2-theory. The morphisms of the theory $\mathrm{End}(I)$ are $\mathrm{Mor}_{\mathrm{End}(I)}(m,n) = \mathrm{Functors}(I^m, I^n)$. The morphisms of the theory $\mathrm{End}(I)^k$ are k-tuples of morphisms of $\mathrm{End}(I)$. For $m \in$ Obj $\mathrm{End}(I)$ the category $\mathrm{End}(X)(m)$ has objects Obj $\mathrm{End}(X)(m) = \coprod_{n\geq 0} \mathrm{Mor}_{\mathrm{End}(I)^k}(m,n)$, in other words, the objects of $\mathrm{End}(X)(m)$ are the arrows of $\mathrm{End}(I)^k$ with domain m. For $\prod_{i=1}^p v_i, \prod_{i=1}^q w_i \in$ Obj $\mathrm{End}(X)(m)$ where

$$v_1, \ldots, v_p, w_1, \ldots, w_q \in \mathrm{Mor}_{\mathrm{End}(I)^k}(m,1)$$

we define the set of morphisms $\mathrm{Mor}_{\mathrm{End}(X)(m)}(\prod_{i=1}^p v_i, \prod_{i=1}^q w_i)$ to be the collection of 2-natural transformations

$$\alpha : X \circ v_1 \times \cdots \times X \circ v_p \Longrightarrow X \circ w_1 \times \cdots \times X \circ w_q.$$

Note that $X \circ v_1 \times \cdots \times X \circ v_p$ and $X \circ w_1 \times \cdots \times X \circ w_q$ are functors $I^m \longrightarrow Cat$. For any morphism $u : \ell \longrightarrow m$ of the theory $\mathrm{End}(I)$, define the functor $\mathrm{End}(X)(u) : \mathrm{End}(X)(m) \longrightarrow \mathrm{End}(X)(\ell)$ on objects by $\mathrm{End}(X)(u)(\prod_{i=1}^p v_i) := \prod_{i=1}^p v_i \circ u^{\times k}$ and similarly on morphisms. Thus, $(\mathrm{End}(X), \mathrm{End}(I))$ is an example of a 2-theory.

Example A.17. We define the 2-theory (Θ, T) of *commutative monoids with cancellation* as follows. Let T be the theory of commutative monoids and let $+ : 2 \longrightarrow 1$ and $0 : 0 \longrightarrow 1$ be the usual words in the theory of commutative monoids. Let $k = 2$. The 2-theory Θ is generated by three words: addition $+$, cancellation $\check{?}$, and unit 0. These are described in terms of a general algebra $X : I^2 \longrightarrow Sets$ over (Θ, T) as follows. Note that $+$ and 0 have two meanings.

$$+ : X_{A,B} \times X_{C,D} \longrightarrow X_{A+C,B+D}$$

$$\check{?} : X_{A+C,B+C} \longrightarrow X_{A,B}$$

$$0 \in X_{0,0}.$$

These generating words must satisfy the following axioms.

1. The word $+$ is *commutative*.

$$\begin{tikzcd} X_{A,B} \times X_{C,D} \arrow[r, "+"] \arrow[d] & X_{A+C,B+D} \arrow[d, equal] \\ X_{C,D} \times X_{A,B} \arrow[r, "+"'] & X_{C+A,D+B} \end{tikzcd}$$

2. The word $+$ is *associative*.

$$\begin{tikzcd}
(X_{A,B} \times X_{C,D}) \times X_{E,F} \arrow[r, "+\times 1_{X_{E,F}}"] \arrow[d] & X_{A+C,B+D} \times X_{E,F} \arrow[d, "+"] \\
X_{A,B} \times (X_{C,D} \times X_{E,F}) \arrow[d, "1_{X_{A,B}}\times +"'] & X_{(A+C)+E,(B+D)+F} \arrow[d, equal] \\
X_{A,B} \times X_{C+E,D+F} \arrow[r, "+"'] & X_{A+(C+E),B+(D+F)}
\end{tikzcd}$$

3. The word $+$ has *unit* $0 \in X_{0,0}$.

$$\begin{tikzcd}
X_{A,B} \times \{0\} \arrow[r, "+"] \arrow[dr, "pr_1"'] & X_{A+0,B+0} \arrow[d, equal] \\
& X_{A,B}
\end{tikzcd}$$

4. The word $\check{?}$ is *transitive*.

$$\begin{tikzcd}
X_{(A+C)+D,(B+C)+D} \arrow[r, "\check{?}"] \arrow[d, equal] & X_{A+C,B+C} \arrow[d, "\check{?}"] \\
X_{A+(C+D),B+(C+D)} \arrow[r, "\check{?}"'] & X_{A,B}
\end{tikzcd}$$

5. The word $\check{?}$ *distributes* over the word $+$.

$$\begin{tikzcd}
X_{A+C,B+C} \times X_{E,F} \arrow[r, "+"] \arrow[dd, "\check{?}\times 1_{X_{E,F}}"'] & X_{(A+C)+E,(B+C)+F} \arrow[d, equal] \\
& X_{(A+E)+C,(B+F)+C} \arrow[d, "\check{?}"] \\
X_{A,B} \times X_{E,F} \arrow[r, "+"'] & X_{A+E,B+F}
\end{tikzcd}$$

6. Trivial cancellation is trivial.

$$\begin{tikzcd}
X_{A+0,B+0} \arrow[r, "\check{?}"] \arrow[dr, equal] & X_{A,B} \arrow[d, "1_{X_{A,B}}"] \\
& X_{A,B}
\end{tikzcd}$$

Like a theory, a 2-theory (Θ, T) has a description in terms of operations on the sets $\Theta(w; w_1, \ldots, w_n)$ and many relations the operations satisfy. These operations and relations are listed in [3].

Definition A.18. Let I be a set, $X : I^k \longrightarrow Sets$ a functor, and (Θ, T) a 2-theory. Then X is a *strict* (Θ, T)*-algebra* if it is equipped with a morphism of 2-theories $(\Theta, T) \longrightarrow (\mathrm{End}(X), \mathrm{End}(I))$.

Definition A.19. Let (Θ, T) be a 2-theory. A *pseudo* (Θ, T)*-algebra over* I^k consists of the following data:

- a small pseudo T-algebra I with structure maps $\Phi_n : T(n) \longrightarrow \mathrm{Functors}(I^n, I)$,
- a strict 2-functor $X : I^k \longrightarrow Cat$,
- set maps $\phi : \Theta(w; w_1, \ldots, w_n) \longrightarrow \mathrm{End}(X)(\Phi(w); \Phi(w_1), \ldots, \Phi(w_n))$, where $\Phi(w)$ means to apply Φ to each component of w to make I^k into the product pseudo T-algebra of k copies of I,
- a coherence iso modification for each operation of 2-theories and these coherence iso modifications satisfy coherence diagrams indexed by the relations of 2-theories.

Example A.20. The worldsheets form a pseudo algebra over the 2-theory of commutative monoids with cancellation as in [3] and [5]. This is the same as saying that the worldsheets form a pseudo commutative monoid with cancellation.

Example A.21. The worldsheets form a pseudo algebra over the 2-theory of Frobenius symmetric monoidal categories, or synonymously they form a pseudo Frobenius symmetric monoidal category. Indeed, this is Theorem 4.1.

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